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ABSTRACT

During the years 1993-96, there has existed an active discussion group entitled "Using Open-Ended Problems in Mathematics" as a part of the scientific program of the Psychology of Mathematics Education (PME) conference. This report contains revised versions of presentations given in the discussion group. Since the PME is an international organization and the presenters were selected from different parts of the world, this book gives a worldwide view of the use of open-ended problems in the mathematics classroom. Most contributors hail from Australia, Japan, and the United Kingdom since in these countries, there is a long tradition of using open-ended problems. There were also presentations from other countries including Finland and Taiwan. Most of the papers concentrate on the realization of mathematics teaching with open-ended problems and give a variety of examples. There are also two critical papers that bring forth some uncomfortable and unasked questions regarding the use of open-ended problems. Papers include: (1) "Introduction To the Concept 'Open-Ended Problem'" (Erkki Pehkonen); (2) "Implementing Open Problem-Solving, Some Pitfalls of Curriculum Development through Assessment in the UK" (Paul Blanc); (3) "On the Open-Ended Nature in Mathematical Problem Posing" (Shuk-kwan S. Leung); (4) "Testing Problem Solving in a High-Stakes Environment" (Barry McCrae and Kaye Stacey); (5) "The Institutionalisation of Open-Ended Investigation: Some Lessons from the U.K. Experience" (Candia Morgan); (6) "Communication and Negotiation through Open-Approach Method" (Nobuhiko Nohda and Hideyo Emori); (7) "Use of Problem Fields as a Method for Educational Change" (Erkki Pehkonen); (8) "On the Use of What-If-Not Strategy for Posing Problem" (Hye Sook Seo); (9) "Learning Mathematics through Exploration of Open-Ended Tasks: Describing the Activity of Classroom Participants" (Peter Sullivan, Dianne Bourke, and Anne Scott); (10) "Teaching Children To Think Mathematically" (Howard Tanner and Sonia Jones); and (11) "Literature on Open-Ended Problems" (Erkki Pehkonen). (Author/ASK)

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Research Report

176

Erkki Pehkonen (ed.)

Use of open-ended problems in mathematics classroom

Department of Teacher Education
University of Helsinki

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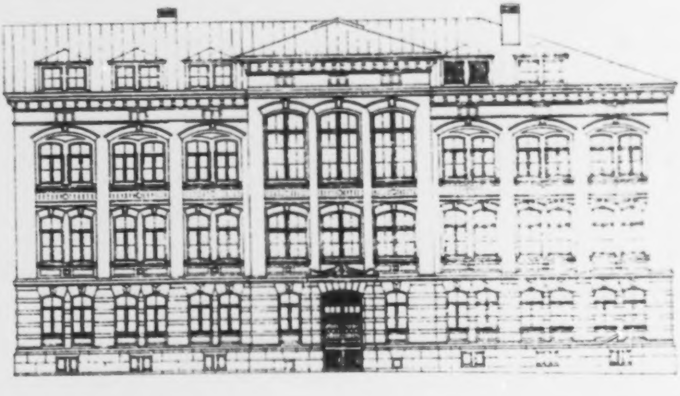

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Research Report 176, 1997

Erkki Pehkonen (ed.)
Use of open-ended problems in mathematics classroom. 130 pp.

Abstract

During the years 1993–96, there has existed an active discussion group "Using Open-ended Problem in Mathematics" as a part of the scientific program of the PME (Psychology of Mathematics Education) conference which discussion group has been organized by the editor. This book contains the revised versions of almost all presentations given in the discussion group.

Since the PME is an international organisation, and the presenters were selected from different parts of the world, the book will give a world-wide look on the use of open-ended problems in mathematics classroom. The most contributors are from three countries: Australia (Dianne Bourke, Barry McCrae, Anne Scott, Kaye Stacey, Peter Sullivan), Japan (Hideyo Emori, Nobuhiko Nohda, Hye Sook Seo) and United Kingdom (Paul Blanc, Sonia Jones, Candia Morgan, Howard Tanner), since in these countries there is a long tradition in using open-ended problems. But there were also presentations from other countries: Finland (Erkki Pehkonen) and Taiwan (Shuk-kwan S. Leung).

Most of the papers are concentrated around the realization of mathematics teaching with open-ended problems, and will give a big variety of examples. But there are also two critical papers (Paul Blanc, and Candia Morgan) which bring forth some uncomformatable and unasked questions about the use of open-ended problems.

Keywords: open-ended problem, investigation, mathematics teaching

HELSINGIN YLIOPISTO
OPETTAJANKOULUTUSLAITOS

Tutkimuksia n:o 176, 1997

Erkki Pehkonen (toim.)
Avoimien tehtävien käyttö matematiikan tunnilla. 130 s.

Tiivistelmä

Vuosina 1993–96 kokoontui aktiivinen keskusteluryhmä "Avoimien tehtävien käyttäminen matematiikan tunnilla" osana PME-kokousten (Psychology of Mathematics Education) tieteellistä ohjelmaa, jonka keskusteluryhmän oli allekirjoittanut organisoinut. Tämä kirja sisältää melkein kaikkien keskusteluryhmässä esitettyjen alustusten korjatut versiot.

Koska PME on kansainvälinen organisaatio, ja alustajat valittiin eri puolilta maailmaa, niin kirja antaa maailmanlaajuisen kuvan avoimien tehtävien käyttämisestä matematiikan tunneilla. Useimmat alustajat olivat seuraavista kolmesta maasta: Australia (Dianne Bourke, Barry McCrae, Anne Scott, Kaye Stacey, Peter Sullivan), Japani (Hideyo Emori, Nobuhiko Nohda, Hye Sook Seo) ja Yhdistyneet kuningaskunnat (Paul Blanc, Sonia Jones, Candia Morgan, Howard Tanner), koska näissä maissa on avoimien tehtävien käyttämisellä pitkä traditio. Mutta mukana oli myös alustuksia muista maista: Suomi (Erkki Pehkonen) ja Taiwan (Shuk-kwan S. Leung).

Useimmat alustukset keskittyvät avoimia tehtäviä käyttävän matematiikan opetuksen ympärille, ja ne antavat laajan valikoiman esimerkkejä. Mutta joukossa on myös kaksi kriittisesti suhtautuvaa artikkelia (Paul Blanc ja Candia Morgan), jotka ottavat esille joitakin epämukavia ja julkilausumattomia kysymyksiä avoimien tehtävien käyttämisestä.

Avainsanat: avoin ongelma, tutkimustehtävä, matematiikan opetus

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Introduction to the concept "open-ended problem"

Erkki Pehkonen
University of Helsinki, Finland

In the conference on Psychology of Mathematics Education (PME-17, Tsukuba, Japan) in July 1993, there was a discussion group on the use of open-ended problems in mathematics classrooms, conducted by the author. For structuring the discussion, there were four brief presentations from different parts of the world. These presentations were elaborated into written form, and published as a theme "Using Open-ended Problems in Mathematics Class." in the journal International Reviews on Mathematical Education (see Pehkonen 1995). In a way, this theme formed a continuation for the earlier theme "Problem Solving in Mathematics" published some years ago (see Pehkonen 1991). The discussion group has its continuation in the PME conferences of three following years (Lisbon 1994, Recife 1995, Valencia 1996).

A brief history of open-ended problems

The method of using open-ended problems in classroom for promoting mathematical discussion, the so-called "open-approach" method, was developed in Japan in the 1970's (Shimada 1977). About at the same time in England, the use of investigations, a kind of open-ended problems, became popular in mathematics teaching (Wiliam 1994), and the idea was spread more by Cockcroft-report (1982). In the 1980's, the idea to use some form of open-ended problems in classroom spread all over the world, and research on its possibilities is very vivid in many countries (e.g. Nohda 1988, Pehkonen 1989, 1995, Silver & Mamona 1989, Williams 1989, Mason 1991, Nohda 1991, 1995, Stacey 1991, 1995, Zimmermann 1991, Clarke & Sullivan 1992, Silver 1993, 1995). In some countries, they use a different name for open-ended problems; for example in the Netherlands, they call their method "realistic mathematics" (Treffers 1991).

The idea of using open-ended problems in school mathematics has been written in some countries in the curriculum, in a form or other. E.g. in the mathematics curriculum for the comprehensive school in Hamburg (Germany), about one fifth of the teaching time is left content-free, in order to encourage the use of mathematical activities (Anon. 1990). In California, they are suggesting open-ended problems to be used in assessment beside the ordinary multiple-choice tests (Anon. 1991). In Australia, some open problems (e.g. investigative projects) are used in the final assessment since the late eighties (Stacey 1995).

Within last years, there have occured also some critical papers on the use of open-ended. A couple of years ago, an American mathematician wrote a sceptical paper on learning mathematics with open-ended problems, or the form of open-ended problems used in Californian schools (Wu 1994). In the Valencia meeting, Paul Blanc critisized very strongly the realization of investigations in British schools (Blanc & Sutherland 1996). He blamed that the teachers have developed a new mechanistic scrutine for solving investigations.

The concept "open problem"

One aim of the PME discussion group in Tsukuba (Japan) was to find answers to the question What are "open-ended problems"? since the group of open-ended problems seemed not to be well disguished. Under the discussion, several types of problems were put forward: *investigations, problem posing, real-life situations, projects, problem fields* (or problem sequences), *problems without question*, and *problem variations* ("what-if"-method). In my conclusion, I suggested the use of the concept "open problem" as an "umbrella" class of problems which contains all the problem classes mentioned.

The concept "open problem" could be explained as follows: We will begin with its opposite, and say that a problem is closed, if its starting situation and goal situation are closed, i.e. exactly explained. If the starting situation and/or the goal situation are open, i.e. they are not closed, we have an open problem (Fig. 1). Problems dealt with in school mathematics are usually closed problems (or more generally closed tasks) which will not leave much room for creative thinking.

According to this definition, we have three types of open problems where one is the open-ended problems. The groups of problems given above provide us with some examples, which are placed into the boxes of Fig. 1. It is worthwhile noting that a group of problems, e.g. real-life situations, may cover several types of open problems.

goal situation starting situation	CLOSED (i.e. exactly explained)	OPEN
CLOSED (i.e. exactly explained)	closed problems	open-ended problems real-life situations investigations problem fields problem variations
OPEN	real-life situations problem variations	real-life situations problem variations projects problem posing

Fig. 1. The classification of problems according to their starting and goal situations.

Some mathematics educators use the word "exploratory" as a synonym to "open" (e.g. Avital 1992), in order to prevent confusion with the unsolved problems of mathematics (cf. also Silver 1995).

The structure of the book

Many examples of open problems can be found in the papers of the book. Most papers are presenting the ideas how to use open problems in mathematics class and the research results of their use. But there are also two critical papers (Paul Blanc, and Candia Morgan) which bring forth some uncomformatable and unasked questions about the use of open-ended problems. At the end of the papers, there is a preliminary bibliography on open-ended problems. The papers in the book are given in the alphabetical order.

References

Anon. 1990. Lehrplan Mathematik. Lehrplanrevision Gesamtschule. Sekundarstufe I. Behörde für Schule, Jugend und Berufsbildung, Amt für Schule. Hamburg.

Anon. 1991. A Sampler of Mathematics Assessments. Sacramento (CA): California Dept of Education.

Avital, S. 1992. Exploratory Problems as a Motivational Factor in the Learning of Mathematics. A paper presented in the Working Group 5 of the ICMI-7 conference in Quebec.

Blanc, P. & Sutherland, R. 1996. Student teachers' approaches to investigative mathematics: iterative engagement or disjointed mechanisms? In: In: Proceedings of the PME-20 conference (eds. L. Puig & A. Gutierrez), Vol. 2, 97–104. Valencia: University of Valencia.

Clarke, D.J. & Sullivan, P.A. 1992. Responses to open-ended tasks in mathematics: characteristics and implications. In: Proceedings of the PME 16 (ed. W. Geeslin & K. Graham). Volume I, 137–144. Durham (NH): University of New Hampshire.

Cockcroft Report 1982. Mathematics counts. Report of the Committee of Inquiry into the Teaching of Mathematics in Schools. London: H.M.S.O.

Mason, J. 1991. Mathematical problem solving: open, closed and exploratory in the UK. International Reviews on Mathematical Education (= ZDM) 23 (1), 14–19.

Nohda, N. 1988. Problem solving using "open-ended problems" in mathematics teaching. In: Problem Solving – A World View (eds. H. Burkhardt, S. Groves, A. Schoenfeld & K. Stacey), 225–234. Nottingham: Shell Centre.

Nohda, N. 1991. Paradigm of the "open-approach" method in mathematics teaching: Focus on mathematical problem solving. International Reviews on Mathematical Education (= ZDM) 23 (2), 32–37.

Nohda, N. 1995. Teaching and Evaluating Using "Open-Ended Problem" in Classroom. International Reviews on Mathematical Education (= ZDM) 27 (2), 57–62.

Pehkonen, E. 1989. Der Umgang mit Problemfeldern im Mathematikunterricht der Sek. I. In: Beiträge zum Mathematikunterricht 1989, 290-293. Verlag Franzbecker, Bad Salzdetfurth.

Pehkonen, E. (ed.) 1991. Problem solving in mathematics. International Reviews on Mathematical Education (= ZDM). Part 1. 23 (1), 1–19. Part 2. 23 (2), 32–50.

Pehkonen, E. 1993. On Teachers' Criteria to Assess Mathematical Activities. In: Proceedings of the seventeenth PME conference (eds. I. Hirabayashi, N. Nohda, K. Shigematsu & F.-L. Lin). Volume I, 220–227. University of Tsukuba, Tsukuba (Japan).

Pehkonen, E. (ed.) 1995. Using Open-ended Problems in Mathematics Class. International Reviews on Mathematical Education (= ZDM) 27 (2), 55–72.

Pehkonen, E. 1997. Use of problem fields as a method for educational change. In this book.

Shimada, S. (ed.) 1977. Open-end approach in arithmetic and mathematics – A new proposal toward teaching improvement. Tokyo: Mizuumishobo. [in Japanise]*

Silver, E.A. 1993. On mathematical problem posing. In: Proceedings of the seventeenth PME conference (eds. I. Hirabayashi, N. Nohda, K. Shigematsu & F.-L. Lin). Vol. I, 66–85.. University of Tsukuba, Tsukuba (Japan).

Silver, E.A. 1995. The Nature and Use of Open Problems in Mathematics Education: Mathematical and Pedagogical Perspectives. International Reviews on Mathematical Education (= ZDM) 27 (2), 62–67.

Silver, E.A. & Mamona, J. 1989. Problem posing by middle school mathematics teachers. In: Proceedings of PME-NA 11 (eds. C.A. Maher, G.A. Goldin & R.B. Davis). Volume 1, 263–269. New Brunswick (NJ): Rutgers University.

Stacey, K. 1991. Linking application and acquisition of mathematical ideas through problem solving. International Reviews on Mathematical Education (= ZDM) 23 (1), 8–14.

Stacey, K. 1995. The Challenges of Keeping Open Problem-Solving Open in School Mathematics. International Reviews on Mathematical Education (= ZDM) 27 (2), 67–72.

Treffers, A. 1991. Realistic mathematics education in The Netherlands 1980–1990. In: Realistic mathematics education in primary school (ed. L. Streefland), 11–20. Utrecht: Freudenthal Institute.

Wiliam, D. 1994. Assessing authentic tasks: alternatives to mark-schemes. Nordic Studies in Mathematics Education 2 (1), 48–68.

Williams, D. 1989. Assessment of open-ended work in the secondary school. In: Evaluation and Assessment in Mathematics Education (ed. D. F. Robitaille), 135–140. Science and Technology Education. Document Series 32. Paris: Unesco.

Zimmermann, B. 1991. Offene Probleme für den Mathematikunterricht und ein Ausblick auf Forschungsfragen. In: International Reviews on Mathematical Education (= ZDM) 23 (2), 38–46.

Wu, H. 1994. The Role of Open-Ended Problems in Mathematics Education. Journal of Mathematical Behavior 13 (1), 115–128.

* There exists a (partial) translation into English which I received from Prof. Jerry P. Becker (Southern Illinois University, Carbondale).

Implementing open problem-solving, some pitfalls of curriculum development through assessment in the UK

Paul Blanc
University of Bristol, U.K.

Motivation

Someone essentially in favour of investigative approaches standing up in front of an audience enthusiastic about the merits of open problem-solving and forecasting doom seems slightly absurd. However, the UK implementation of investigative approaches to mathematics has been fraught with difficulties, the vast majority of which remain unresolved. Can we, as an international community, learn from the experience of other countries? Clearly there will be great differences in social and cultural context, in curricula, and in the nature of the mathematics studied, but it is not unreasonable to assume that some of the difficulties and phenomena encountered in the UK are likely to emerge in other countries represented in the Open Problem-Solving Discussion Group in PME. The experience of Australia would appear to confirm this (Stacey 94). In this article we discuss changes to the UK curriculum regarding investigative, problem-solving approaches to mathematics, *how* these changes have been effected and *what* some of the consequences have been.

Open problem solving and investigations

Whilst this is not the place for a long discussion on the nature of open problem-solving, nonetheless we wish to acknowledge that the terms open problem-solving, investigative work, and their variants are problematic. We will adopt Morgan's (95) notion of three discourses for investigative work; the *'official'* (Government reports and exam board publications), the *'professional'* (mathematics education research literature on theoretical and practical issues) and the *'practical'* (text books, teachers' guides, guides on how to do coursework), and affirm that since there is no essentially true reading of curriculum texts, these discourses should not be regarded as a hierarchy.

Problem-solving itself ranges from applying standard techniques in routine exercises, to thinking creatively about some situation (often not explicitly mathematical). Exploring any patterns involved, posing problems and seeking solutions using formulation, testing and proof of conjectures are also aspects of problem-solving. This type of mathematical activity finds its roots in Polya's problem-solving strategies (Polya 57) and has been developed in work on mathematical thinking by, for example, Mason et al (85). The United Kingdom has over the past 15 years undergone significant curriculum change where an investigative approach to mathematical problem-solving has been a major issue.

Changes to the UK Curriculum

In the 1980s, largely supported by the mathematics education community, UK official publications promoted changes to the curriculum, suggesting a move towards mathematics as an exploratory, dynamic, evolving discipline and recommending the use of open problem-solving and investigations (Cockcroft Report, DES 82). The subsequent changes in the UK led to problem-solving and investigations becoming part of the official mathematics curriculum. Problem-solving (op cit. p73) is defined as the ability to apply mathematics to a variety of situations, to everyday situations experienced by the child as well as the unfamiliar. The role of substantial discussion in the move from the problem statement to any form of writing is emphasised. Investigational work is expressly stated as not consisting solely of long projects (extended pieces of work) but as "perhaps most frequently" emanating from pupils' questions. Pupils should try to find the answer for themselves. Teachers should encourage a willingness to ask; "what would happen if?" The theme of discussion is renewed; "Much of the value of an investigation can be lost unless the outcome of the investigation is discussed" (op cit. p 74). It is interesting that the notion of write-up or formal written communication is not present.

The new initiatives stated in the Cockcroft report were implemented via: the GCSE (examination at age 16), the new curricular schemes produced, notably by the School Mathematics Project (SMP 11-16); 350 mathematics advisory teachers were appointed to disseminate 'good practice'; graded assessment schemes, aimed mainly at low attainers

(Brown, 1996). One should not underestimate the influence of the first two in this list. The SMP scheme was eventually to be found as a resource in 70% of schools and the GCSE introduced coursework, to be compulsory by 1991.

Though inclusion of problem-solving and investigations as assessed coursework for GCSE has ensured implementation in UK schools, motivation for actually adopting an investigative approach varied. For some schools this became a significant part of the pedagogy with a group of teachers committed to this approach. However, more often, the new innovation would be dealt with in separate lessons to prepare pupils for the oncoming examination (Cooper 94). Whilst some schools adopted the coursework voluntarily, many waited until they had no choice. Ninety percent of teachers put off its introduction for at least a year and most delayed as long as possible, introducing only a minimum 20% weighting in 1991.

Assessment driven curriculum change.

Burkhardt (88) supports the view that assessment and testing effects actual "implemented" curriculum change - the practice in classrooms - stressing that exams are more powerful than exhortation. He suggests that in the past, the influence of public examinations has been negative, narrowing the pattern of learning activities and reinforcing the over-emphasis on narrow technical skills. Since "What you test is what you get" (WYTIWYG), the test should be made valuable in itself, encouraging a range of approaches to mathematics, including investigative tasks. He cites the GCSE as a brave attempt with less than adequate support. The slow uptake both of investigative approaches and of optional coursework in GCSE would appear to reinforce these comments.

Whilst the GCSE has forced the hand of teachers, there is a significant mismatch between the apparently consensus view (from official and professional sources) that investigative approaches are desirable and how these approaches have been operationalised in GCSE maths. In particular, since the GCSE had to be examinable and standardised the focus has been on the written script of the investigation, thus taking away much of the value of discussion emphasised in Cockcroft. The practical consequences of examined coursework have contributed to students seeing investigations as isolated projects, not as an open, creative approach to mathematics, but as an artefact. In writing the script the student is writing to *solve* the problem, and writing to *communicate* the solution (thus providing evidence for assessment). Morgan's study (95) suggests that much of the practical and official discourse emphasises the latter. Although Mason (85) argues that subsequent writing-up with an audience in mind has a role in developing mathematical thinking, the nature of classroom activity, particularly at GCSE coursework time, has focused on directing students through a number of stages to produce the final product. The structuring of investigations, some done in silent examination conditions, can be seen in Figure 1 written for 16+ examination by a leading examination syndicate.

There are clear messages for teacher and student as to what is an appropriate mathematical response through the explicit structuring of the students' activity. Specific questions are to be answered at the start of the problem. Some openness is provided but the way to tackle the problem is specified, via a checklist. Next we find another phenomenon of investigational work: the inevitable "extension", which has become a routine part of the official and practical discourse (Morgan 95, p398).

The specific practice takes you away from the real learning.

Since teachers and students lack experience and confidence in this way of working, considerable advice has been put forward by examination boards, through guides to coursework and in curriculum materials. This has evoked a tension between openness and creativity on one hand and examples, advice, and assessment schemes on the other. The practical discourse concerned itself with listing "desirable properties of a piece of coursework", ostensibly to help teachers and pupils. Consequently pupils have a set of approved recipes, at least at the meta-level, and this relatively explicit information, together with pupils' awareness that "what you do is what is expected" (WYDIWIE) can lead to students fulfilling this 'contract' rather than engaging with the mathematics. As mentioned before, advice/mark schemes frequently concentrate on the form of communication (especially presentation of results) therefore students frequently focus on this.

Figure 1

TRIANGLES IN POLYGONS

Triangles are formed by joining the vertices of a polygon. The vertices of the triangles must be vertices of the polygon.

Fig 1

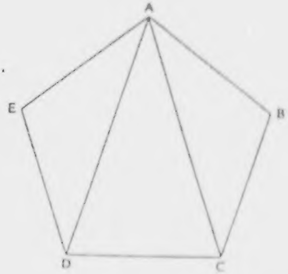
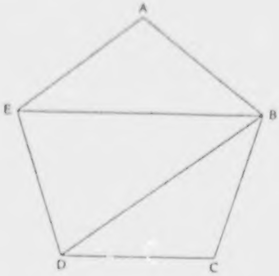


Fig 2



Part 1

a) The regular pentagon shown in figures 1 and 2 shows six different triangles.
Show that by joining C to A and C to E it is possible to make *only two* new triangles.

b) Show that it is possible to form 10 triangles, each of which has 3 of the vertices of the pentagon as its vertices.

Part 2

For regular polygons investigate the relationship between the number of triangles and the number of vertices of the polygon.

Where appropriate:
- use diagrams;
- record your observations and results;
- make conjectures and hypotheses;
- generalise your results;
- comment on any exceptions or counter examples and offer any explanations.

Part 3 (EXTENSION)

Either classify the various triangles according to their properties or extend the problem in some other way.

There has recently been a debate about the effectiveness of such mathematical investigations in the UK. Hewitt (92) has questioned whether the diversity and richness of such open ended problems is being reduced to spotting patterns from tables. Wells (93) in a contentious pamphlet introduces the notion of Data-Pattern-Generalisation (DPG) as a general routinary method of solving problems said to have little or very limited mathematical value. The overall thrust of these arguments is that the potential positive advance of pupils exploring mathematics at their own level has been seriously undermined by the algorithmic and mechanical nature of the approaches adopted by pupils in schools. Barnard & Saunders (94) also maintain that an instrumental understanding of content is being replaced by an instrumental understanding of process. Success is obtained through learning "the rules of the game", not the mathematics. Students are learning how to behave in a given scenario (or frame) where the meta-level sub-tasks, such as: Do the Examples, Collect the Information in a Table, Find the Rule represents mathematical investigation. Hewitt (92) demonstrates how a wide range of starting points *can* be reduced to pattern spotting from tables, essentially the same activity which has its own limited intrinsic value (if desired, it can be related to finite difference methods, which fit given data to functions). It would appear that this activity has moved a long way from the intentions of curriculum innovation, seeing through the examples, having an in-depth look at special cases to examine similarities and differences, looking for the generic in the particular, seeking the underlying structure of the problems, exploring the mathematics.

Explanations?

We seek to understand these outcomes by considering the dynamics of the teacher-student relationships within an institutionalised educational setting. Some of the more negative effects of the use of open problem-solving could be explained by Brousseau's (86) notion of the meta-cognitive shift in which perceived failure on the part of students can lead to the teacher imposing heuristics as objects of study instead of the mathematics intended. Brousseau suggests that this phenomenon is more likely when heuristics, advice and models are given the status of cultural objects and he uses Venn diagrams in the "modern mathematics" movement as an example of this effect (Brousseau 86). This phenomenon is not due to inadequacies on the part of teachers and pupils, but is in fact an inevitable (or at least potential) consequence of any teaching situation. The institutionalisation of meta-level guidance on how to approach open problem-solving can readily be seen in UK curriculum materials. For example in materials produced by the Shell Centre (84, p.46) the following key strategies are recommended: Try some simple cases; Find a helpful diagram; Organise systematically; Make a table; Spot patterns; Use the patterns; Find a General Rule, Explain why it works; Check regularly. There are many such instances throughout the practical discourse of investigations.

One major difficulty in analysing the overall effects of the changes in practice is that much of the evidence relies on the reflection of respected and valued individuals, gained though experience of, and interaction between communities of teachers, learners, teacher educators, official bodies and so on. Unfortunately, within the professional discourse, debate has seemed to polarise. Criticism of specific practice in investigations has sometimes been seen as criticism of investigative approaches or as direct criticism of teachers and support for traditional emphasis on skills learnt through text books. Advocates of investigative approaches have sometimes contrasted 'believers' with 'non-believers', fearing a backlash. More recently some studies have sought to examine exactly what mathematics students' are engaged with when tackling particular investigative, open problems (for example, Blanc and Sutherland 1996).

The international mathematics education community has sometimes accepted the value and desirability of open problem-solving, if only greater teacher participation could be ensured. The case of the UK raises a number of issues which should be taken into account in curriculum development which seeks to develop such participation.

- open problem-solving is ill defined and resists definition;
- any reform is highly problematic (even if there is apparent consensus);
- texts do not communicate intentions in an unproblematic way;
- assessment driven curricula may increase quantity of open problem-solving but alters its nature;
- openness and creativity sit uncomfortably with standardised exams;
- students' activity may become highly structured through explicit problem statements;
- students' attention may focus on meta-level guidance rather than the mathematics itself;

- students may routinise tasks, specific open problem-solving practices may evolve;
- students focus on written outcomes and presentation of results.

In the final sections we examine two contrasting research projects in the light of the initial discussion. Firstly, Askew's examination of teachers' practice in implementing aspects of more recent curriculum reform. Secondly, a small scale study into how some students interact with a specific problem. Many of the phenomena above are reflected in these studies.

National Curriculum Reform

Ironically the increased politicisation of the curriculum swung the curriculum pendulum back towards assessment and league tables for schools comprised in the main of formal tests, concentrating on skills. Coursework is no longer compulsory in GCSE (from 1995), though examinations must still assess process aspects of mathematics, now classified under the heading of "Using and Applying Mathematics" in the National Curriculum. This was a separate part of the UK National Curriculum which resulted, albeit unintentionally in a separation of process from content. Askew reports on a large scale study in teachers' practical discourse of this new innovation (Askew 96).

The development of "Using and Applying Mathematics".

"Using and Applying Mathematics" formalised process aspects of mathematics. In the first version of the official document, the idea that this should "permeate all other work in mathematics, providing both the means to and the rationale for, the progressive development of knowledge skills and understanding in mathematics" was relegated to a non-statutory appendix! (DES 1989). Following concerns stimulated by government commissioned reports (National Curriculum Council 91) that this area of mathematics was not receiving sufficient attention, the second version stated:

> "Pupils should choose and make use of knowledge, skills and understanding in the programmes of study in practical tasks, in real-life problems and to investigate within mathematics itself. Pupils would be expected to use with confidence the appropriate mathematical content

specified in the programmes of study relating to the other attainment targets."
(DES/WO 91 p1).

Askew's study examines why teachers were finding difficulty in integrating "Using and Applying Mathematics" into their work. Results from a questionnaire to 744 schools highlighted lack of teaching experience in "Using and Applying Mathematics", inadequate coverage by commercial schemes, lack of clarity of meaning in official documents and difficulties related to need for changes in teaching style, but curiously the majority of responses did **not** (as in the official reports) regard coverage of "Using and Applying Mathematics" as problematic (op cit.). Again this highlights a mismatch between official discourse and specific practice. Askew reports, through analysis of interview data how teachers were adopting a limited perspective on "Using and Applying Mathematics", concentrating on what they saw as *'relevant mathematics'* (e.g. money, using mathematical equipment) *'practical mathematics'* (measurement, real-events) and for later years *'stand-alone investigations'* (banks of bolt-on tasks mainly involving algebraic generalisation). One dominant approach linked the task to matching all pupil abilities in a desire to offer something for everyone. This would appear to confirm Brousseau's 'Topaze' effect: tasks are made progressively easier until students can produce (at least some) required behaviour without engaging in the mathematics of the task (Brousseau 86).

Although analysis of questionnaires indicated high degree of planning, interview data presented a different picture. Most teachers indicated that it was not necessary to plan specifically for "Using and Applying Mathematics" giving reasons, for example, that all maths had these elements or that they used these ideas in an opportunistic manner. Despite difficulties experienced in introducing investigative problem-solving via the GCSE, the intentional fallacy: that any reading of texts corresponds to authors intentions, re-emerges in Askew's study. The intentions of curriculum developers would seem to remain distant from the experience of most children in the mathematics classroom. The texts themselves do not help matters. The paradox of giving illustrative examples is that they can become the sole practice. Askew comments that 8 out of the first 9 examples involve measurement, hence identifying "Using and Applying Mathematics" with the *'practical mathematics'* reported by teachers above in his study.

The notion of a permeation model is also suspect. How exactly does this work? Some examples indicate that content can be developed through problem solving but few teachers had spent much time on this part of documentation. In crude terms many teachers had concentrated on the title "Using and Applying Mathematics", regarding this as real-world applications only. Askew points out a stark contrast between his findings and interpretations offered in professional discourse. Mathematics educators may wish to read the documents to suit their own purposes, as a means to develop content in an investigative manner, as well as more generally problem-solving abilities such as the "understanding of, and ability to engage in, reasoning, logic and proof, the processes of mathematical argument and justification" (Askew 96 p111). However, there is considerable evidence that this interpretation is far from the readings or the practice of most teachers. Askew's conclusions demonstrate that curriculum change in this area has been partial and fragmentary and that teachers may be encouraged to believe that "Using and Applying Mathematics" can be implemented with minimal changes to existing 'traditional' practice.

Thus far we have discussed some shortcomings of curricula reform and some concerns as to the implementation of open problem-solving in GCSE mathematics and the National Curriculum. Now we consider a study specifically concerned with how students interact with a particular problem, which is part of a larger project examining students' uses of external representations and heuristics in investigative problem-solving (Blanc and Sutherland 95). In this study a major concern was the effect of meta-level guidance given to students. Could we identify students who were creatively engaged in mathematics as opposed to adopting mechanistic approaches? If so, would this shed light on the causes of specific problem solving behaviour?

Creative engagement or disjointed mechanisms?

Small-scale in-depth studies can probe more precisely what students are gaining from specific tasks in specific contexts. In recent research, the relative influences of task itself, form of the task, student engagement, previous experience, knowledge level , heuristics known and external representations used were examined. Blanc and Sutherland (96) discuss the analysis of written solutions to the question, *'How many diagonals does a polygon have?'*, for a group of 14 first year primary school trainee teachers. This cohort included mature students and recent school leavers. These two groups have had very different experiences of school mathematics because of the recent changes in UK curricula. Using protocol analysis, verifying interpretations through semi-structured interviews we were able to ascertain some interesting features of students' ways of working, the selection of external representations employed and the nature of students solutions. Interestingly most students used annotated diagrams, tabular collection of information, algebra and natural language as external representations. We found that it was not whether a particular device was employed that mattered but how this device fitted in to the overall solution which differentiated the outcomes. One student made use of her graphical representations directly from her own constructions, suggesting that constructing diagrams for yourself as opposed to being presented with a static constructed diagram could make a difference to the problem-solving process. Tabular representations were used in an inflexible way by 3 of the 5 students studied in depth. Two of these stated that they had received strong advice about doing tables and the third said she had been strongly influenced by her partner to draw a table. In these three cases the table seems to act as a **separator** so that work after the table uses only the table itself as a potential source of information. This supports the contention of decontextualised pattern spotting (Wells 93) and may be due to inflexible use of some taught process model of problem-solving. Tables were used however in a highly flexible, dynamic way by two students (Blanc & Sutherland 1996). Crucially, of the students who were most successful in solving the problem, their engagement was the overriding concern. External representations or heuristics were only useful to them in so far as they helped them continue to solve the problem, less successful students seemed to use the representations as mechanisms.

Some students work in a linear fashion down the page with little reworking or looking back. This led in some cases to discontinuities in solution and failure to exploit potentially crucial information. In contrast, others' attention moves all over the work, that is, they use their written text in a non-linear way using a varied range of representations in an **iterative** manner adjusting, correcting and enhancing.

In relation to the previous discussion these results indicate that meta-level guidance and routines were evident in tackling a particular investigation. Those students who did not adopt these routines were apparently more engaged and more successful in the problem itself. The empirical work also demonstrated that analysis of the texts alone can identify routinised pattern spotting behaviour erroneously. This adds a further caveat to the dangers of narrowing the task through examinations which assess only a written product. Further, the act of producing a text to communicate a solution (which will be read as a linear script), may focus students on the communication of results, in a pre-defined, inflexible order when the interaction of their different external representations may help them progress both with the problem and in their mathematics generally. These issues provide the foci of continuing research.

Concluding remarks.

This article highlights how curricula reform in open problem solving has affected the work of both teachers and students but not necessarily in the manner expected. In both cases the intentions of the reformers have been only partially realised. Changes have been resisted and delayed. Implementation has been fragmentary and the readings of implementers are some way from the intentions of the developers (Askew 96). Whilst assessment has increased the occurrence of investigative work, the nature of such work has been dramatically influenced by standardised assessment. Some consequences include generalised guidance on 'how to do' investigative mathematics, viewing investigations as projects, routinised approaches, highly structured tasks and a focus on written outcomes at the expense of discussion. On a more specific level we have reported some evidence that students do use guidance in a routine and unproductive manner, sometimes appealing to specifically taught strategies as justification of their actions. There is also evidence of more flexible problem solvers engaged not with the representations or meta-level guidance but with the problem.

The studies discussed share one common feature: that looking in more depth (in both cases through analysis of interview data) revealed how surface interpretations, firstly of teachers' responses and secondly of students' written scripts can be misleading. Whilst we do not pretend to have quick fix solutions for the task of implementing investigative, open problem-solving, we would wish to stress that the issues raised are not solely attributable to UK specific conditions. In fact, many of the consequences would appear to be relevant to any country intending similar reforms.

References

Askew, M. (1996) Using and Applying Mathematics in Schools: Reading the texts, in Johnson D, and Millet, A. (eds.) *Implementing the Mathematics National Curriculum*, Paul Chapman, London.

Barnard, T. & Saunders, P. (1994) *Superior sums that don't add up to much*. The Guardian Newspaper 28/12/95.

Blanc, P. & Sutherland, R. (1996) Student Teachers' Approaches To Investigative Mathematics: Iterative Engagement Or Disjointed Mechanisms?, *Proc. 20th Int. Conf. Psychology of Mathematics Education, Vol. 2, pp 97-104*, Valencia, Spain.

Blanc, P. (1995) Investigating The Mathematics Learning Of Student Teachers: Explorations And Discoveries?. *Proceedings of British Society for Research in Learning Mathematics* Conference, 3/95, University of Birmingham.

Brousseau, G. (1986) Fondations et méthodes de la didactique des mathématiques. *Recherches en didactique des mathématiques, 7/2, p33-115*.

Brown, M. (1996) The Context of the Research-The Evolution of the National Curriculum for Mathematics in Johnson D, and Millet, A. (eds.) *Implementing the Mathematics National Curriculum*, Paul Chapman, London.

Burkhardt, H. (1988) National Testing, Liability or Asset?, *Mathematics Teaching 122, pp33-35*

Cockcroft, W. (Chair) (1982) *Mathematics Counts*, Report of the Committee of Enquiry into the teaching of Mathematics in Schools. London: HMSO

Cooper, B.(1994) Secondary Mathematics Education in England: recent changes and their historical context, in Selinger, M. (ed.) *Teaching Mathematics*, Routledge, London.

DES/WO (1989) *Mathematics in the National Curriculum, Non-Statutory Guidance*. London, HMSO.

DES/WO (1991) *Mathematics in the National Curriculum*, Department of Education and Science /Welsh Office, London.

Hewitt, D. (1992) Train Spotters' Paradise, *Mathematics Teaching* 140, 6-8.

Mason, J. (1994) Assessing what sense pupils make of mathematics, in Selinger, M. (ed.) *Teaching Mathematics*, Routledge, London.

Mason, J. Burton, L. & Stacey, K. (1985) *Thinking Mathematically* Wokingham: Addison Wesley.

Morgan, C. (1995) *An analysis of the discourse of written reports of investigative work in GCSE Mathematics*, PhD thesis, Institute of Education, University of London.

National Curriculum Council (1991) Report on Monitoring the Implementation of the National Curriculum

Polya, G. (1957) *How to Solve It*. Princeton: Oxford University Press.

Shell Centre. (1984) *Problems with Patterns and Number, An O level module*, Joint Matriculation Board, Manchester.

Stacey, K (1994) *The challenges of keeping open problem-solving open in school mathematics*, Preprint 4/94, Dept Science and Mathematics Education, University of Melbourne, Australia.

Wells, D. (1993) *Problem Solving and Investigations*, Bristol: Rain Press.

On the Open-ended Nature in Mathematical Problem Posing

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Summary
Problem solving is divided into two stages: Representation and Solution. In Representation stage, a problem is well-structured if the objects, operators and goals are well defined. Ill-structure problems are problems with undefined objects or goals. They open up opportunities for problem posing. However, well-structured problems can be posed again to serve instructional purposes. In this paper, the author provided specific examples on problem posing research for four cases: defined/undefined cross object/goal. These examples were used to discuss the open-ended nature of mathematical problem posing; especially when problem posing is followed by problem solving.

Problem solving, according to cognitive scientists, is generally divided into two stages: Representation and Solution. In Representation stage, a cognitive structure corresponding to a problem is constructed by a solver on the basis of his domain-related knowledge and its organization (Chi, Feltovich, & Glaser, 1981). In the Solution stage, a person considers paths from the initial state to the goal state but the solution depends on how the person represents the given problem in the first stage.

The Structure of a Problem

A problem is well-structured if the objects, operators and goals are well defined (Reitman, 1965). Reitman identified four categories of problems according to how well the given and goal states are specified:

- Undefined given and Undefined goal state
- Undefined given and Well-defined goal state
- Well-defined given and Undefined goal state
- Well-defined given and Well-defined goal state

Based on the above specifications, only problems in the last category are well-structured. In the first three categories, there are undefined components. The word "undefined" also included "ill-defined" cases. Therefore, the first three categories are known as ill-structured problems. Ill-structured problems open up opportunities for problem posing; the problem "poser" has to define the given, the goal or both.

A relationship between ill-structured problems and problem finding was found in Voss (1990). Indeed, "problem finding, that is, how individuals formulate and identify a problem ... in itself is an ill-structured problem" (Voss, 1990, p. 12). The reasons were two. First, the representation of the problem can vary according to how the person perceives the situation. Second, there are no generally agreed upon solutions (in this case, solutions are actually problems). Reitman's classification and Voss' discussion suggested that ill-structured problem solving included problem posing, and in terms of defining the given, the goal, or both. Finally, problem posing is also possible, even in well structured problem situations. In this paper, the author provided specific examples on problem posing research for all four cases of Reitman and discussed the open-ended nature of mathematical problem posing; when problem posing is followed by problem solving.

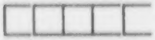
Problem Posing by Attending to Reitman's Classification

Undefined Given and Undefined Goal State. In Reitman's first category, when the given and the goal are not specified, the problem posing activity is most open. Subjects were asked to pose problems without any reference to constraints supplied by the task environment. For example, U. S. children were asked to pose story problems in class (Winograd, 1991), while Australian children were told to make a problem that is difficult for a friend to solve (Ellerton, 1986). In these instances, the problem posers were free to make variations in story situations as well as in mathematical structures. Winograd found that children pose a variety of story problems using real life experiences, while Ellerton reported that children planned ahead of the complexity in mathematics structures (e.g. making problems complicated initially but allowing canceling of fractions to make answers simple).

In one study, student teachers were asked to pose a sequence of problems for a mathematics test during student practice (Leung, In press). The activity in posing problems for a mathematics test is also open-ended. There is no limitation on the number of test items, the mathematics topic, the difficulty levels, the order of the items, nor the format of test items (multiple choice, true-false, fill-in-blank, or free format). It was found that student teachers tended to construct well-structured problems. For example, when the test is on multiplication or division of fractions, the items were of a variety in multiplicative structures (see Vergnaud, 1983). Empirical example problems in *Isomorphism-Of-Measures* were posed: "A book is 3\f(1,2) cm thick. If 5 are piled together, what is the total thickness?" (*Multiplication*); "Several balls weigh 2\f(2,3) kg. If each weighs \f(1,3) kg, how many balls are there?" (*Quotitive division*); "Half a pizza is divided equally among three children, how much did one receive?" (*Partitive division*); and "A steel rod of 1\f(1,2) m long weighs 3 kg. What is the length of a steel rod weighing 1 kg? weighing 5 kg?" (*Rule-of-three*). In addition, these student teachers asked questions at all five levels identified in a chapter on posing problems properly by Butts (1980). The five levels are recognition, algorithm, application, open search and problem situation (Leung, In press).

Undefined Given and Well-defined Goal State. In this category, the given is undefined but not the goal. Therefore, the one who poses a problem has to think of possible givens and a reasonable problem that would lead to the goal. For example, when the answer provided for elementary school children to pose a problem is "Speed of A=37m/min; Speed of B=28 m/min" a problem posed by a child was "The runway is 1038 m long. A finishes in 28 min. B finishes in 37 min. Find the speed of A and B" (Leung, July 1996). Another example is seen in a workshop conducted on Complex Instruction (Cohen, 1993). The well-defined goal is "a circle" and participants were to find the possible sets of "givens" (cut-out parts of a circle), which when put together, will result in a complete circle. It is noted that the problem posing activity is open in that many possible problems can be posed accordingly. For example, in the speed problem, a person with more mathematics background can change the typical distance-time-speed perspective and pose a problem that involves simultaneous equations, "The speeds of two trains are A and B. Find the two speeds if the sum is 65 and the difference is 9". Also, the activity in resulting a complete circle as the goal can be changed to a problem on graphing the locus of a point which involves the equation of a circle. The open nature in this kind of problem posing, is on both defining the given and on matching a problem to arrive at a well-defined goal.

Well-defined Given and Undefined Goal State. The above two cases were instances where the given is undefined. When the given is well-defined, the problem posing activity is open in terms of defining the goal. For example, prospective elementary school teachers posed problems by referring to given information in a story situation (see Leung, 1993). They were able to pose a variety of arithmetic word problems in various semantic structures. Occasionally, they posed incomplete problems, insufficient problems or implausible problems.

In another study, instead or verbal clues, the given is a pictorial clue, like  , a matchstick arrangement (Silver & Leung, 1992). Subjects were able to use the given matchstick arrangement and posed, problems that were for the the most part relating to the number of sticks and the number of squares. However, the subjects also posed problems that did not belong to the above category, such as "please use these sticks to make a pattern that is symmetrical". They could even add their own imagination to the original pattern. For example, one subject marked "A" to one end of the incomplete square and "B" to another end and the problem attached was, "Find all possible ways a person can walk from A to B".

In another study, the given situations were more restricted than the above two examples, and in most cases only one particular problem "naturally follows" (Leung and Jou, 1995). The design of the task resembled that of "problem with unstated question" by Krutetskii (1976). The investigators developed problem situations on multiplication and division for grade four children to pose problems. For example, if the well-defined given is "Tea are packed in half kg cans. There are 28 cans altogether, ___________? " then most children asked "what is the total weight of all 28 cans?" From this example, one can see that there is very little openness in the activity of finding the natural question that follows, when compared to that of Leung (1993) and Silver and Leung (1992).

Sometimes, a given situation may also provoke the generation of situation-related questions as well as mathematical problems. One example is the Billiard Ball Mathematics task surveyed by Silver, Mamona, Leung and Kenny (1996). The task environment described a ball shot at an angle of 45 degrees, from the lower left hand corner of a rectangular billiard table which also and rebounds at an angle of 45 degrees. The instructions did not direct subjects to pose only mathematical problems. Silver et al found that subjects asked mathematical questions like ,"what is the number of hits when the table is a square?", and also, questions related to Billiard Ball playing such as, "what happens if you hit the ball too hard?" In this example, the openness was on the nature of the problems that were posed.

Well-defined Given and Well-defined Goal. The above three cases indicated how ill-structured problems can be converted to well structured problems via problem posing. However, there can also be problem posing even when the given and goal are stated explicitly. This could happen when one poses a problem for others to solve. In doing so, problem posing is a technique of representing a known problem differently. For example, the investigators promoted problem posing by supplying a mathematical computation (e.g. Greer & McCann, 1991) and requiring subjects to write a story that matched the computation.

In posing problems that originate from a well-structured problem, the openness is on mathematics content, mathematics representation, context building, and questioning techniques (Leung, 1996). Here, the given and the goal are both inherent; what is open is how to "pose problems properly" (Butts, 1980). The job is often an exercise of teachers and of textbook developers. For example, if the well-structured problem is one on Subtraction-with-regrouping: 30-18=( ). This well structured problem can be posed differently by attending to the above four factors. In mathematics content, one can attend to semantic structures (CHANGE, COMBINE, COMPARE) and posed a CHANGE problem ("Amy has 30 dollars in her purse, she spent 18, how many are left?") or a COMPARE problem ("Amy has 30 dollars and Bob has 18; how much more dollars does Amy have?"). Next, in mathematics representation, one can use algorithms (vertical, horizontal), table (showing different pairs of numbers), photographs or pictures (of purses and money), words or combinations of the above examples to present a given problem. In addition to content and representation, the attention can be switched to context building. The above subtraction problem can be embedded in a problem with reference to quantities other than money (such as length, weights, volumes, ... etc.). The contexts of a problem can be real, imaginary, in-school or out of school. However, familiar contexts are found to be more comprehensible to students (Van den Heuvel-Panhuizen; 1996). Finally, the questioning of a given problem can be "how many dollars left?", "can you tell us how you found the answer?" or "Connie's answer is 18 dollars, do you think she is correct? Why?" Hence, a well structured problem that is presented for solution, which is ready for solving, can be posed again in order to obtain at a certain instructional effect.

Problem Solving, Problem Posing and Open-ended Learning

Since the publication by the National Council of Teachers of mathematics of the Agenda for Action, which asserted that the acquisition of problem solving skills should be one of the goals of school mathematics introduction in the 1980s, problem solving has been a dominant topic at virtually all professional meetings of mathematics teachers and supervisors. Rarely in the history of education has a topic simultaneously captured so much of the attention of both researchers and practitioners (Silver, 1985, p. vii). Contemporary reform efforts not only place a heavy emphasis on problem solving but also on problem posing. The suggestions in both the Curriculum and Teaching Standards (NCTM, 1989; 1991) imply that problem posing is an integral part of problem solving and should not be emphasized separately from problem solving.

Given the inseparable nature of the two activities, a discussion on the open-ended nature of mathematical problem posing should be completed by considering problem solving as well. Even in Polya's book, *How To Solve It*, Polya suggested that teachers ask students to pose problems that they are going to solve (Polya, 1945). Polya's four phase model in problem solving is: Understand, Plan, Carry out, and, Look back. This is true when a person solves a problem given by others. If the problem is not already given, the goal is then to formulate a problem and then to solve it. Polya's problem solving model will become a cycle for problem posing and problem solving. Figure 1 shows this enhanced problem posing-solving cycle.

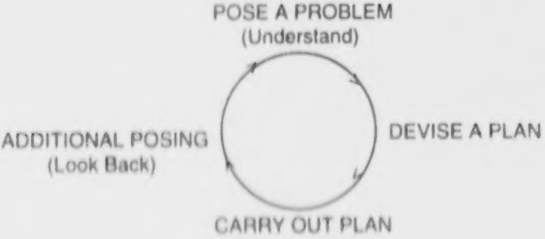


Figure 1. Enhanced Model of the Problem Posing-Solving Cycle.

In the first phase, "Understand the problem" can be replaced by "Pose a problem". If one solves the problem he or she poses, the understanding phase is already included in the problem posing phase. Then the person plans and carries out the plan. After solving, the "Look back" stage is represented by evaluating the solution and posing more problems. Thus, a problem posing and solving cycle exists. The four phase model of Polya will become "Pose-Plan-Carry Out-Pose". Through this problem posing and solving cycle, a person poses and solves problems according to his/her own perspectives and "make mathematics", in Polya's terms. This learner's problem posing and solving activity is open in both mathematics contents and in learners' interaction with mathematics (Nohda, 1987). It is open also in representation and extensions in exploration (Silver & Mamona, 1990). In addition to openness, the above figure also shows that the Pose-and-solve activity can be never-ending thus, depicts and embarks a wonderful, prominent open-ended nature in mathematical problem posing and solving.

Note: This paper includes a summary of contents orally reported at a discussion group in PME 18 (Lisbon, Portugal, 1994) and an afterthought on the open-ended nature of problem posing activity. It is hoped that issues raised in this paper will promote the thinking of the role of open-ended activities in mathematics instruction.

Reference

Butts, T. (1980). Posing problems properly. In Stephen Krulik and Robert E. Reys (Eds.). *Problem solving in school mathematics* (pp. 23-33). Reston, Virginia: National Council of Teachers of Mathematics.

Cohen, E. (June, 1993). *Complex Instruction*. Workshop conducted at national Chang Hwa Normal University.

Chi, M. T., Feltovich, P., & Glaser, R. (1981). *Categorization and representation of physics problems by experts and by novices*. Pittsburgh, PA: Learning Research and Development Center.

Greer, B., & McCann, M. (1991). Children's word problems matching multiplication and division calculations. In F. Furinghetti (Ed.), *Proceedings of the fifteenth annual meeting of the International Group for the Psychology of Mathematics Education* (pp. 80-87). Assisi, Italy: Author.

Ellerton, N. F. (1986). Children's made up mathematics problems--a new perspective on talented mathematicians. *Educational Studies in Mathematics, 17*, 261-271.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in school children.* Chicago: University of Chicago Press.

Leung, S. S. (1993). Mathematical problem posing: The influence of task formats, mathematics knowledge, and creative thinking. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F. Lin (Eds.). *Proceedings of the 17th International conference of the International Group for the Psychology of Mathematics Education, Vol. III* (pp. 33-40). Tsukuba, Japan: Author.

Leung, S. S. , & Jou, S. Y. (1995). The posing and solving of multiplication and division problems by grade four children : A preliminary study. *Proceedings of 10th Annual Conference on Mathematics and Science Education* (pp. 401-414), National Taiwan Normal University. (In Chinese)

Leung S. S. (July, 1996). *On the Role of Creative Thinking in Problem Posing.* Invited Speech for Topic Group 7, 8th International Congress of Mathematics Education (ICME-8), Seville, Spain.

Leung, S. S. (1996). Assessing mathematics textbooks through problem posing: An example of subtraction with re-grouping. *Elementary Teacher Journal, 37* (4), 23-28. (In Chinese)

Leung, S. S. (In press). Problem posing as assessment: Reflections and reconstructions. *The Mathematics Educator.*

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

Nohda, N. (1987). A study of open-approach method in school mathematics. *Tsukuba Journal of Educational Study in Mathematics,* 5, 119-131.

Polya, G. (1945). *How to solve it* (2nd ed.). New York: Doubleday.

Reitman, W. (1965). *Cognition and thought.* New York: Wiley.

Silver, E. A. (Ed.). (1985). *Teaching and learning mathematical problem solving: Multiple research perspectives.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Silver, E. A., & Leung, S. S. (1992). *Mathematical problem posing and problem solving: An investigation of processes, products, and relationships.* Paper presented at the annual meeting of American Education Research Association, 1992, San Francisco.

Silver, E. A., & Mamona, J. (1990). Stimulating problem posing in mathematics instruction. In G. Blume & M. K. Heid (Eds.), *Implementing new curriculum and evaluation standards.* University park, PA: Pennsylvania Council of Teachers of Mathematics.

Silver, E. A., Mamona-Downs, J., Leung, S. S., & Kenny, P.A. (1996). Posing mathematical problems in a complex task environment: An exploratory study. *Journal for Research in Mathematics Education, 27* (3), 293-309.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127-174). New York: Academic Press.

Van den Heuvel-Panhuizen, Marja (1996). *Assessments and realistic mathematics education.* Utrecht, The Netherlands: CD-b Press, Center for Science and Mathematics Education, Freudenthal Institute.

Voss, J. F. (1990). Das Losen schlecht struckturierter Probleme - ein uberblick. (On the solving of ill-structured problems: A review). *Unterrichts Wissenschaft, 18,* 313-337.

Winograd, K. (1991, April). *Writing, solving and sharing original math story problems: Case studies of fifth grade children's cognitive behavior.* Paper presented at the annual meeting of the American Educational Research Association, Chicago.

Testing problem solving in a high-stakes environment

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Summary
Assessing problem solving as part of students' final public assessment at the end of school is a very effective way of encouraging schools, teachers and students to give problem solving a serious place in the curriculum, but can genuine problem solving survive the pressure? We report the ways in which one system of high-stakes problem solving assessment has adapted to demands for public credibility, reliability of judgement, and practicality and efficiency for teachers and students. The success of the resulting tasks is evaluated in terms of the extent to which students have to demonstrate problem solving which is creative and open and deal with problems which are unfamilair in techniques required or context. We also report results from a questionnaire seeking teachers' evaluation of the success of assessing problem solving by a combination task and test.

Introduction

It is a commonplace that one of the most effective way to change practices in schools is to change the method of assessment. If something is assessed, then it is more likely to be taught by teachers and learned by students. In the late eighties, the mathematics subjects taught to students in the last two years of school in Victoria, a state of Australia, were substantially redesigned and a new credential, the VCE (Victorian Certificate of Education), replaced the former Higher School Certificate. It is important to note that this credential is very important to students, their parents and their schools because it is the principal measure that determines students' access to limited higher education places and it is informally used by many as a measure of school quality.

The designers of the new VCE mathematics subjects wished to encourage in students the development of problem solving skills, a capacity for mathematical modelling and the ability to independently undertake mathematical investigations. They therefore made it compulsory for all students to spend at least 20% of their mathematics time on independent mathematical projects and at least 20% on problem-solving and modelling, which is defined as 'the creative application of mathematical skills and knowledge to solve problems in unfamiliar situations, including real-life situations' (Board of Studies, 1996, p. 7). In addition, as part of the final assessment of their level of performance (with consequences for tertiary entrance places, etc), all students had to complete an *Investigative project* and a *Challenging problem* to assess their problem-solving and modelling ability.

An earlier paper, presented to the Discussion Group on Open Problem Solving and later published by Stacey (1995) included a description of the assessment of the *Investigative project.* It demonstrated how the project changed over several years in order to make it easier to organise in schools and to make the assessment more reliable. It was noted there how pressures from teachers and students had lead to a reduction in the openness of the project. Encouragement for the student to make significant decisions about the direction of the investigation had gradually been replaced by a series of questions to be answered with some degree of openness, but much less than in earlier years. Expectations that students would seek out their own data had been replaced by the provision of data sets from which students select. The expectation that students extend or generalise their work in a way of their own choosing was replaced by suggestions for a direction for extension and strong guidance about the intended scope of any extension. Although these changes made the project evolve into a substantially altered task, the judgement was that the end result was still a valid assessment of investigative work in mathematics.

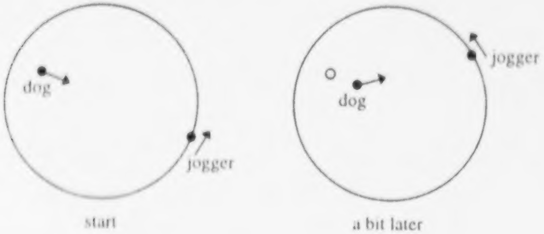
In the present paper, we summarise the rather different history of the *Challenging problem* and its successor, the *Problem-solving task*. This story illustrates the clash of the ideals of having students undertake in-depth problem solving with the practicalities of the classroom and the constraints of a large examination system, subject to sometimes intense public scrutiny. In describing the various compromises that have been made we are concerned to explore the central question of whether real problem solving can survive the constraints of a high-stakes assessment system.

From Challenging problem to Problem-solving task: 1989–1996

For the first four years of the VCE, from 1989 until 1992, one of the major assessment tasks in each of the mathematics subjects was the*Challenging problem.* Students were given two weeks to solve one of three or four problems and to prepare a written report of up to 1000 words on their solution. All students in the state chose from the same set of problems, which were written each year by a centrally appointed committee of mathematicians and teachers. Each problem was a 'real' problem, either pure or applied in nature, which had not been thoroughly mathematised. Figure 1 shows one of the 1989 problems.

Dog and Jogger

A jogger runs around a circular track at a constant speed. The jogger's dog has wandered off and the jogger whistles it to come back. The dog runs at a constant speed towards the jogger. The dog's direction changes as the jogger continues running around the track.



- If the dog starts inside the circular track and runs more slowly than the jogger, what path will the dog follow?
- What path will the dog follow if it runs more quickly than the jogger? What if the dog and the jogger run at the same speed?
- What if the dog starts outside the track?
- How do your results compare with the behaviour you would expect of a real dog in the situations described above?

Figure 1. An early Challenging problem.

Schools graded their own students' reports, according to a common set of criteria, but these assessments were subject to a process of verification involving teachers from other schools. Initially, the task was designed to be, as far as possible, a genuine problem solving experience. It was expected that students might discuss their thinking with others, as any real problem solver might, but that any help received would be acknowledged so that it could be taken into account when a grade was awarded. Some problems (such as the Dog and Jogger) were even identified as being suitable for group work. The report that students wrote had to describe their solution to the problem and also show evidence of reflection on the process of problem solving that they had experienced.

After considerable adverse publicity, the *Challenging problem* was suspended by the authorities at the end of 1992 because of growing concerns that many students were being unfairly advantaged in the preparation of their reports by assistance from parents, tutors or friends, even if this assistance was acknowledged in the reports. Rumours circulated that solutions to the problems were being bought on the streets, causing high anxiety in candidates and parents. The concerns over authentication of students' reports that confronted the *Challenging problem*, and the development of a revised means of assessing problem solving which addressed these concerns, are discussed more fully in Stephens and McCrae (1995).

From 1994, a two-component *Problem-solving task* has replaced the original *Challenging problem*. In the first part of the assessment, students prepare an 800–1200 word report on one of three centrally-set problems over a period of two weeks. Group work is no longer allowed. The second part is a test, conducted under examination conditions shortly after the due date for completion of the report, which requires the students to solve a related, but not identical, problem. The test is designed to provide evidence of the authenticity of students' reports, by showing whether the students understand the mathematics they used in their report of solving the original problem. Figure 2 shows one of the 1994 problems and Figure 3 shows the related test.

> **Problem 1, 1994: The Art Gallery**
>
> The mathematical techniques which might be required for this task include:
> trigonometry—compound-angle formulas, inverse trigonometric functions
> calculus—differentiation using quotient rule, maximisation
> functions—domain, range
>
> While any other prescribed methods are acceptable, the above are considered particularly appropriate, and will feature in the test which will follow this task.
>
> **Question 1**
>
> A room in an art gallery contains a picture which you are interested in viewing. The picture is two metres high and is hanging so that the bottom of the picture is one metre above your eye level. How far from the wall on which the picture is hanging should you stand so that the angle of vision occupied by the picture is a maximum? What is this maximum angle?
>
> **Question 2**
>
> On the opposite wall there is another equally interesting picture which is only one metre high and which is also hanging with its base one metre above eye level, directly opposite the first picture. Where should you stand to maximise your angle of vision of this second picture?

Question 3
How much advantage would a person 20 centimetres taller than you have in viewing these two pictures?

Question 4
This particular art gallery room is six metres wide. A gallery guide of the same height as you wishes to place a viewing stand one metre high in a fixed position which provides the best opportunity for viewing both pictures simply by turning around. The guide decides that this could best be done by finding the position where the sum of the two angles of vision is greatest. Show that the maximum value which can be obtained by this approach does not give a suitable position for the viewing stand.

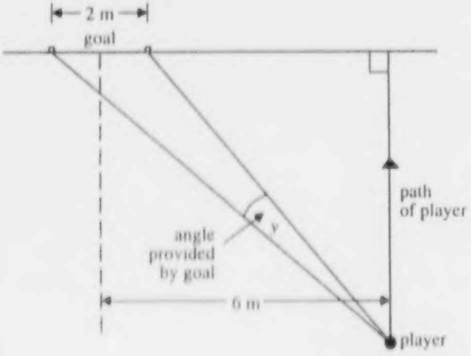
Question 5
The gallery guide then decides to adopt an alternative approach which makes the difference between the angles of vision of the two pictures, when viewed from the viewing stand, as small as possible. Where should the viewing stand be placed using this approach? Comment on your answer.

Figure 2. The problem in a recent Problem Solving task.

The report and the test are both graded by the student's school using guidelines provided by the examination authority, the Victorian Board of Studies, and are subject to a statewide review process. The final grade is obtained by combining the report and the test grades in a 60:40 ratio. If the grade of the report is much higher than the grade on the test, the school must interview the student to investigate the authenticity of the student's report. If the student does not convince the interview panel that he/she understands the content of the report, the grade of the report is reduced to the grade of the test. Disciplinary procedures are implemented if it appears that the student was not the author of the report. No action is taken if the grade on the test is higher than the grade for the report.

Test 1, 1994: Ice hockey
Ice hockey is a team game played on an ice rink, in which players try to get the puck (a rubber disc) into the opposition goal. The goals are actually six feet in width (1.83 metres), but to simplify the arithmetic in this problem, they shall be assumed to be two metres wide.

Question 1
A player is moving in a straight line perpendicular to the line of the goal and 6 metres away from a parallel line through the centre of the goal (see diagram below).
When the player is 10 metres from the goal line, what is the angle which the goal provides for aiming at as seen from the player's point of view? (This is angle y in the diagram.)



Question 2
If the distance of the player from the goal line is x metres (rather than 10 m), find an expression for the tangent of the angle which the goal provides. By differentiating this expression with respect to x, or otherwise, show that this angle never exceeds 10 degrees. Explain why it is sufficient to differentiate tan y with respect to x rather than having to find dy/dx.

Question 3
Within what range of values of x must the player shoot in order that the angle provided by the goal (angle y) is greater than, or equal to, 8∞?

Figure 3. The test related to The Art Gallery problem.

Is problem solving alive and well in the VCE?

The main purpose of this paper is to discuss the extent to which the problem solving task as now constituted does indeed assess problem solving and modelling, as defined in the official documents for the VCE. This will be done by reporting results of a questionnaire sent to teachers after the first implementation of the new structure in 1994 and by analysing the problems set that year: The Art Gallery, Gaussian Integers and The Last Lap. The second problem presents the definition of Gaussian integers (complex numbers of the form $m + ni$ where m and n are ordinary integers) and primes and asks a series of questions about the relationship between Gaussian primes and ordinary primes. In The Last Lap, the speeds of two runners on the last 400m lap of a race are given as functions of time (one linear, one trigonometric) and a series of questions are asked concerning the relative positions and speeds of the runners during the lap.

The teacher questionnaire was sent to all of the 24 schools which had participated in a trial of the format held in 1993 (the year before full scale implementation) and to another 108 schools selected at random from the approximately four hundred schools who offered candidates in 1994. Completed questionnaires were received from 15 (62.5%) of the first group of schools and from 65 (60.2%) of the second group; an overall response rate of 60.6% involving 953 (19.2%) of the 4963 students who completed the *Problem-solving task* .

The questionnaire contained 39 questions covering eight areas: background information, problem attributes, report structure, test attributes, comparing report and test grades, interviews, authentication issues and other issues. Most questions required respondents to select one of five responses representing positions along a continuum of possible opinions (e.g. strongly disagree, disagree, not sure, agree, strongly agree). In addition to the 39 questions, common incorrect answers to some of the test questions were given and the respondents were asked to mark them according to the marking scheme provided by the Board of Studies. There was also space at the end of the questionnaire to add comments and 62 (65%) of the 80 respondents commented further on at least one aspect. The results of this questionnaire, which are reported more fully by McCrae (1995), have been used to improve the design of the problems and tests and refine the regulations.

As noted earlier in this paper, the definition of problem solving given in the VCE mathematics study design booklet is 'the creative application of mathematical skills and knowledge to solve problems in unfamiliar situations, including real-life situations' (Board of Studies, 1996, p. 7). In discussing whether problem solving has survived the transition in method of assessment from *Challenging problem* to *Problem-solving task*, it is fruitful to consider how this definition of problem solving has been operationalised.

Creativity and openness

A solution to a problem might be called 'creative' if some aspects of the question(s) being answered have been invented by the solver or if the method of solution is left open to the solver. A comparison of the problem statements in Figures 1 and 2 shows the way in which the openness of problem solving has been lost over the years. In the Dog and

Jogger problem the students are given no hint as to how to proceed, other than that three distinct situations should be considered, whereas in The Art Gallery their work is organised around a sequence of closed questions each requiring a specific mathematical response.

The questionnaire revealed that teachers generally supported the format of a sequence of closed questions with 77.9% of respondents indicating that this should be a feature of future *Problem-solving tasks*. There was general agreement amongst the teachers that a problem structured in this way is more accessible (more students find it easier to start) and less time-consuming for students, and is easier to assess reliably, than a more open *Challenging problem* style problem. However, only about a third of respondents felt that the closed question format increased their confidence in the authenticity of their students' solutions.

Did the teachers, although they supported the use of closed questions, believe that this changed what was being assessed? In fact, 48.7% of the respondents agreed that the closed problem format *reduced* its validity as a measure of problem solving ability, with 30.8% disagreeing and 20.5% undecided. It seems clear that teachers as a whole have been prepared to sacrifice some validity in the assessment of problem solving for practical convenience and concern for their students' results.

One of the recommendations made to the Board of Studies concerning the future conduct of the *Problem-solving task* as a result of the questionnaire was that each problem should involve some opportunity to generalise solutions to improve the validity of the task as a measure of problem solving ability. There has been an attempt to do this ever since as can be seen by contrasting the questions in Figure 2 with the two examples, one from 1995 and one from 1996, shown in Figures 4 and 5. (Note that both of these problems consisted of a graded sequence of questions. Only the last question is reproduced in each case.) The Oil Pipelines question (Figure 4) is more open in that the solver needs to determine how to proceed; the Disappearing Wombats question is more a straightforward exercise in algebra.

Problem 1 (1995): Oil Pipelines

An oil platform is situation at a point O, at sea, 10 km from the nearest point P on a stretch of straight coastline. An oil refinery is at a point R, 16 km along the coast from the point P. It is necessary to lay a pipeline from the platform to the refinery. This is to consist of straight line sections. Laying pipeline underwater is 4/3 times as expensive as laying pipeline on land. This ratio is known as the *relative cost*.

...

Question 5

Another platform is to be located at a point Q, 8 km from shore on the other side of R, opposite a point Y, 4 km from R (see diagram below). Explore whether there are any cost advantages in combining pipelines, stating any assumptions you have made.

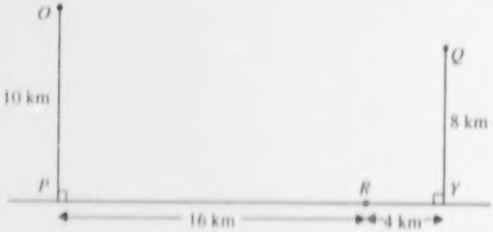


Figure 4. Recent problem demonstrating opportunity to generalise.

Problem 3: Disappearing Wombats

Scientists are concerned that a species of wombat may be in danger of extinction, as low numbers have been observed in the forests where they live. In an effort to protect the species from extinction, the Department of Conservation and Natural Resources decided to trap 200 wombats and move them to a remote island off the Australian coast. It is hoped that the wombat population will recover there safe from the problems on the mainland.

...

A more sophisticated mathematical model for the number W of wombats on the island, t years after the initial 200 are settled there, takes into account the availability of wombat food. Under this model, we have

$$\backslash f(dW,dt) = (m - n - kW)W$$

where m and n are related to the birth rate and death rate respectively, and k is a positive constant related to the amount of food available on the island.

...

Question 5

Suppose $m = 0.10$, $n = 0.06$ and $0 < k \leq 0.001$.

a. Find expressions for the number of wombats on the island after t years for some carefully chosen values of k and sketch corresponding graphs showing how the wombat population changes over time.

b. According to the model, what will happen to the wombat population eventually, for each of your chosen values of k?

c. Generalise your answer to part **b.** for any positive constants m, n and k, such that $m > n$.

Figure 5. Recent problem demonstrating opportunity to generalise.

Creativity and familiarity

Another aspect of the meaning of creative problem solving is related to the way in which creativity requires a person to combine ideas in new ways. This is reinforced in the VCE definition in that the problems are required to be unfamiliar. Unfamiliarity has at least two aspects: a problem may be 'unfamiliar' because it involves an unfamiliar context (the real situation in which it is set) or it may be unfamiliar because it is a non-routine problem—i.e. it is not amenable to a taught solution method (such as solution by simultaneous equations) at least without considerable work beforehand.

Both of these aspects of creative problem solving were probed in the questionnaire to teachers. Teachers were asked to indicate on a scale of 1 to 5 how familiar each 1994 problem was to their students, with a high rating indicating that students had never before encountered a similar problem. Teachers considered that Gaussian Integers was very unfamiliar (average rating 4.0) whereas they considered that students had encountered problems similar to The Art Gallery or The Last Lap before (average rating 3.2 for both).

A similar pattern was observed when we asked teachers to rate, from the students' point of view, the type of mathematical strategies required to solve the problems on a scale ranging from routine (score 1) to creative (score 5). The Art Gallery and The Last Lap both scored in the middle of the scale at 2.9 and 2.8 respectively, whereas teachers rated Gaussian Integers as requiring significantly more creative strategies (average score 4.3).

It therefore appears that teachers regarded The Art Gallery and The Last Lap as less valid measures of problem solving ability than Gaussian Integers. Perhaps surprisingly this was not the case. On a rating scale from 'not suitable' (rating 1) for students to demonstrate their problem solving abilities to 'highly suitable' (rating 5), teachers gave The Art Gallery and The Last Lap average ratings of 3.8 and 3.7, whereas Gaussian Integers with an average rating of 2.9 was slightly below 'moderately suitable'.

The questionnaire items did not directly help us understand why this was the case. We can only speculate what additional characteristics teachers are looking for in creative problem solving. We believe, from our own observations of teacher constructed tasks in other contexts, that the best explanation is that teachers look for some application to the real world (which they saw in the two favoured problems but not in Gaussian Integers) as a key feature of problem solving tasks. One questionnaire respondent commented in the free response section that problems such as The Art Gallery and The Last Lap are better than 'one which is very obscure and which causes students stress', whilst another respondent wrote that Gaussian Integers was 'a waste of time [as it was] seen as threatening by students'.

Creativity and degree of transfer

Underlying any decision to have creative problem solving tested as an outcome of a program of instruction is the relationship between what is to be learned in the program and what is to be tested. One might argue, for example, that it is grossly unfair for a student's success in mastering any program of instruction to be measured by their performance on items which are unfamiliar and non-routine. What should be the degree of transfer between the sort of problems and situations that would have typically been encountered by students in their study of VCE mathematics and the problems which are used to test their problem solving ability?

The earliest discussions of how problem solving might be assessed canvassed a variety of options. Instead of testing the use of mathematics to solve problems, the ability to carry out or to identify specific problem solving strategies (such as the Polya-type strategy 'look for a similar problem') might have been assessed. Alternatively, given the well-established influence of metacognitive factors on problem solving, students might have been assessed on the extent to which they could demonstrate awareness of aspects of the problem solving process.

In fact, from 1989 to 1994 the reports prepared by VCE students on their problem solution were required to show 'how the student started off, how understanding of the problem developed, and any other important insights and breakthroughs which occurred as they sought a solution to the problem' (Board of Studies, 1995, p. 9). Our questionnaire showed, however, that by 1994 over 50% of students ignored this requirement and, although there was a large spread of opinion, teachers on average rated this requirement as only sometimes being important in assessing a student's problem solving ability. The requirement was dropped for 1995 and has not been reinstated.

A comparison of the 1989 problem shown in Figure 1 with the 1994 problems shows that the mathematical techniques required to solve the 1994 problems are more closely related to those which are taught in the course. To solve the Dog and Jogger problem requires knowledge of vectors and differential equations that was not part of the course at that time and would have had to have been taught as additional material. Nowadays, not only must the problems be set so that additional material is not required for their solution, but each problem statement begins with a list of the mathematical techniques which are most relevant to its solution (see Figure 2).

The relationship between the problem and the test in the presently constituted *Problem-solving task* is the second place where degree of transfer is an issue. The test's validity as an authenticity measure, and the policy of combining the report and test marks to give a single assessment of problem solving ability, rest on this foundation. Early advice on the nature of the test (Stephens, 1994, p.14) had described it as a 'transfer task' and clearly teachers had expected that only a low level of transfer was intended because 91% of the questionnaire respondents expected that, compared with the corresponding problem, the test questions would be sufficiently similar to the problem questions to be able to be answered by each student *using the techniques he/she used in solving the problem.* Further, 50% of these teachers (i.e. 45.6% of respondents overall) expected the test to be set in the same context as the corresponding problem—no doubt influenced by the fact that the problem and the test had the same context in the sample *Problem-solving task* distributed earlier in the year by the authorities.

In 1994, it turned out, however, that only the test for Gaussian Integers satisfied the teachers' expectations. The tests for The Art Gallery and The Last Lap were both set in a different context to the corresponding problem, which teachers described as a particular disadvantage to students for whom English is a second language. In addition, depending on the approach taken to solving the problem, some of the questions on these two tests required students to use techniques that they may not have used in their own solution. Not surprisingly, therefore, both tests were frequently criticised by questionnaire respondents.

In the questionnaire, we sought teachers' opinion on whether the inclusion of the test/interview process had improved various aspects of the validity and credibility of the assessment. We asked if they believed the combined assessment was more valid, whether the amount of cheating would be reduced, whether teachers could better identify cheating and whether the credibility of the assessment was improved amongst the student population and in the view of the public. On the 5 point scale from strongly disagree (1) to strongly agree (5), the average response was slightly over 3 (unsure) for the first three questions and about 4 (agree) for the two questions concerning credibility. Despite the teachers' uncertainty, in the eyes of the Board of Studies the new arrangements have been highly successful in improving the assessment and, from 1997, a similar test/interview process has been added to the *Investigative project* assessment.

Discussion

The introduction of the test requirement into the assessment of problem solving has had a constraining effect on the openness of the problems. If the problems set for the report are at all open, students can solve them in many ways, and some of these ways will be more suitable to answering the questions on the tests than others. The use of technology in The Last Lap problem provides one such example. According to their teachers, students doing this problem depended heavily on technology (graphics calculators or computer spreadsheets) which was not permitted in the test.

The inclusion of a 'generalisation' requirement in most problems since the first implementation of the *Problem-solving task* in 1994 has only redressed one aspect of the loss of openness from the original *Challenging problems*. Of remaining concern is that the presentation of each problem as a sequence of closed questions has removed from the solver most of the responsibility to mathematise the situation. When a real situation is presented and a problem is posed and students are to provide a solution to the problem using any techniques that they can, as was the case with the *Challenging Problems*, there may be many paths the student can take. Different assumptions made in the mathematical models and different variables regarded as highly significant lead to substantially different solutions.

Too much mathematising of the situation which is presented as a problem also means that the discussion of assumptions and the evaluation and interpretation of results, always regarded as important criteria in the overall assessment of problem solving, can be stilted and unnatural. Particular attention needs to be given to the way in which these components of problem solving can be fostered and elicited in the presentation of problems. For example, in the presentation of The Art Gallery problem, the decision to use the size of the angle of vision as a criterion for a suitable viewing place is inherent in the question. There does not seem to be any encouragement in the text to alert students to the need to either consider or discuss whether this is a good measure.

Allowing students two weeks to solve their chosen problem and write their report, which was the case for the *Challenging problem* and is also the case for the *Problem-solving task*, is a recognition that creative problem solving takes time but it was the root cause of the authenticity concerns that lead to the demise of the *Challenging problem*. The introduction of the test/interview process has reduced these concerns by providing public evidence as to whether the student understands what is in the report and (to a lesser extent) whether he or she is its author. However, the feature that distinguishes the best problem solvers is the 'one per cent inspiration' which produces the important insight and breakthrough and the authorship of *this* can only be guaranteed under test conditions. This is an aspect of the assessment of problem solving that has always bothered VCE teachers and that it continues to do so was revealed by our questionnaire. Respondents considered the requirement that students must submit with their final report any draft material and a bound logbook containing all working notes to be much more important than the test/interview process in enabling teachers to feel confident about authenticating students' reports.

References

Board of Studies. (1996). *Mathematics study design (1997–2000).* Carlton, Victoria: Author.

Board of Studies. (1995). *VCE official sample CATs 1995: Specialist Mathematics.* North Blackburn, Victoria: HarperSchools.

McCrae, B. (1995). An evaluation of system-wide assessment of problem solving at year 12 by report and related test. In B. Atweh & S. Flavel (Eds.), *Proceedings of the 18th Annual Conference of the Mathematical Education Research Group of Australasia* (pp. 409–415). Darwin, Northern Territory, Australia: Mathematical Education Research Group of Australasia.

Stacey, K. (1995). The challenge of keeping open problem-solving open in school mathematics. *Zentralblatt für Didaktik der Mathematik 95*(2), 62–67.

Stephens, M. (1994). Developing a problem solving assessment task in the VCE. *Vinculum, 31*(1), 7-17. Melbourne, Victoria: Mathematical Association of Victoria.

Stephens, M. and McCrae, B. (1995). Assessing problem solving in a school system: principles to practice. *Australian Senior Mathematics Journal, 9* (1), 11-28.

The institutionalisation of open-ended investigation: some lessons from the U.K. experience

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Over the past fifteen or so years, 'investigation'[1] has become an official part of the mathematics curriculum in England and Wales. There is, however, a significant degree of contestation about the nature of this activity and its place within the curriculum. Indeed, some would argue that the official endorsement and consequent institutionalisation of investigation by government reports and curriculum and assessment bodies has distorted the nature of the activity to the extent that it no longer addresses the ideals that gave birth to it (Ernest, 1991; Lerman, 1989; Morgan, 1996) . In this paper I intend to provide an overview of the development of investigation within the mathematics curriculum and to raise some questions about the possibility of creating a school curriculum that values open-ended problem solving and creative thinking.

Any attempt to define investigation and to trace its presence in the UK curriculum inevitably encounters difficulties because of the very openness of the phenomenon and the multiple ways in which it has been interpreted. A search of the index of the journal *Mathematics Teaching* of the Association of Teachers of Mathematics (ATM) reveals articles listed under 'investigation' or its cognates dating from 1959, including six separate articles in 1968 (compared to seven in 1982 and six in 1990). This does not mean that teachers and pupils in 1959 or 1968 thought of themselves as 'doing investigations'; rather, the compilers of the index saw similarities between the activities described and their own contemporary (1992) concept of investigation. The widespread use of the term to describe a type of activity within the school mathematics curriculum dates to the early 1980s and, in particular, to the publication of the Cockcroft report (Cockcroft, 1982). This also marked the start of the process of institutionalisation and consequent transformation of the investigation phenomenon that will be described in this paper.

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[1] I use inverted commas here to indicate that the construct 'investigation' is both multi-faceted and contested. Elsewhere in this paper I have omitted the inverted commas. The reader should bear in mind, however, that the word is never unproblematic.

An analysis of the discourse associated with investigation (Morgan, 1995) finds a high degree of consensus among mathematics educators and official curriculum bodies about the desired properties of investigation in the school mathematics curriculum, although, as we shall see, there is less agreement about what sorts of activities might qualify. In principle, investigation is:

- like doing 'real mathematics' in some way that, by implication, other types of school mathematics are not - in particular, it is not practice of skills or reproduction of standard solutions;
- exploratory, open, creative, and 'empowering' for students, having possibly multiple valid outcomes rather than a single correct answer;
- part of 'good classroom practice'.

In practice, the kinds of activities that may be labelled investigation in the mathematics classroom vary substantially, including:

- projects in which students pose and work on their own problems from a given unstructured starting point;

  e.g. *This is a number block.*

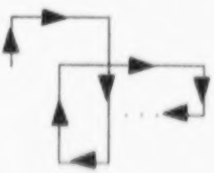
7	*4*	*11*	*15*	*26*

  Starting with 7 and 4, work out how the rest of the number block is filled in. Explore other number blocks.

  (Adapted from Hunt, Huyton, & Taylor (1988).)
- structured problems in which students are guided through a sequence of gathering data, spotting patterns in the data and forming generalisations;
- a general 'investigative approach' during everyday classroom work that encourages posing and following up 'what if . . . ?' questions.

Equally, the mathematical subject matter of investigation can vary. It may be directly related to the traditional content of the curriculum and, with teacher guidance, may lead, for example, to the 'discovery' of Pythagoras Theorem. Alternatively, investigation may start from a situation that appears only tangentially related to the rest of the school mathematics curriculum,

e.g. *Worms leave tracks in layers of mud.*



> *A worm forms a piece of track, turns through 90°, forms another piece, turns, forms another piece, turns, . . .*
> *The drawing shows a 1, 2, 3 worm; it makes one unit of track, turns, makes two units of track, turns, makes three units of track, turns, makes one unit of track, . . .*
> *Investigate worms.* (ATM, 1988)

An investigation of the latter type, while probably making use of knowledge and skills drawn from within the mathematics curriculum, may also proceed in divergent and possibly unpredictable ways that make it impossible to slot into a specific section of the 'content' syllabus.

The general acceptance of the principle that effecting change in the curriculum must be supported (and possibly led) by assessment procedures (Burkhardt, 1988; Ridgway & Schoenfeld, 1994) means that there is also a general agreement that investigation ought to be assessed. I shall argue, however, that there are tensions between the desired properties of investigation and the requirements of an official system of assessment.

The Cockcroft Report - defining official 'good classroom practice'

The report of the Committee of Inquiry into the Teaching of Mathematics in Primary and Secondary Schools in England and Wales (Cockcroft, 1982) is probably best remembered for its much quoted paragraph 243:

> Mathematics teaching at all levels should include opportunities for
> - exposition by the teacher;
> - discussion between teacher and pupils and between pupils themselves;
> - appropriate practical work;
> - consolidation and practice of fundamental skills and routines;
> - problem solving, including the application of mathematics to everyday situations;
> - investigational work. (p.71)

While acknowledging that the ideas contained in this list were by no means new, the report went on to claim that:

> although there are some classrooms in which the teaching includes, as a matter of course, all the elements that we have listed, there are still many in which the mathematics teaching does not include even a majority of these elements.

The report thus constructed a need for change in mathematics classrooms[2], specifically a need to introduce new ways of working. Significantly, a later section of the report makes a connection between this discussion of teaching styles and the form of examination at 16+. Having castigated timed written examinations for causing a state of affairs in which "practical and investigational work finds no place in day-by-day work in mathematics" (p.161), the conclusion is drawn that:

> Because, in our view, assessment procedures in public examinations should be such as to encourage good classroom practice, **we believe that provision should be made for an element of teacher assessment to be included in the examination of pupils of all levels of attainment.**
>
> (p.162, original emphasis)

A clear identification is being made between teacher assessment in public examinations and 'good classroom practice', including investigational work. This part of the report, in particular the recommendation emphasised by the use of bold type in the extract above, may be seen as instrumental in the eventual institutionalisation of investigation as part of the new General Certificate of Secondary Education (GCSE) examination to be taken by all students at 16+.

Within the Cockcroft Report, however, there is a certain ambiguity about the nature of the activity. The range of grammatical forms used to refer to it encapsulates this ambiguity. Firstly, the item "investigational work" in the list in paragraph 243, by qualifying the everyday activity 'work', suggests that this too is just an everyday activity. Indeed, the only exemplification offered stresses its routine nature within the mathematics classroom:

> At the most fundamental level, and perhaps most frequently, [investigation] should start in response to pupils' questions . . . . The essential condition for work of this kind is that the teacher must be willing to pursue the matter when a pupil asks "could we have done the same thing with three other numbers?" or "what would happen if . . ." . . . . Even practice in routine skills can sometimes, with benefit, be carried out in investigational form; for example, 'make up three subtraction sums which have 473 as their answer'.[3] (p.74)

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[2] I use the word constructed advisedly. The report presents no evidence either of the usefulness of the styles it recommends or of the extent of current use of various teaching styles in classrooms.

[3] The ATM response to the final example offered in this extract illustrates some of the contention about the nature of investigation. Fielker claims that this is not investigational because the students'

At the same time, however, the report refers to "investigation" - an abstract object with rather less everyday connotations, perhaps related to scientific activity (or police work). Most significantly, it also refers to the specific object "an investigation". The use of the article suggests that this object, unlike "investigational work" can be separated out from other activities within the classroom and can be clearly identified as an activity in its own right. Whereas the incidental classroom approach or questioning attitude exemplified in investigational work as described above could hardly form part of a formal examination, defining a separate activity as "an investigation" objectifies the nebulous approach into a discrete and assessable form.

The examination of investigation

An early example of the instruction to 'investigate' is found in two questions from an Associated Examining Board examination paper of 1966:

> 4 [*After a definition and an example of a simple continued fraction*] Investigate simple continued fractions.
>
> 5 Investigate either: Quadrilaterals: classification by symmetry, or: Triangles and their associated circles.(AEB, 1966, cited by Wiliam (1994))[4]

Without access to marking schemes or students' scripts it is not possible to say what sorts of responses were expected or how they would have been evaluated. As we shall see, however, the apparent lack of structure in the questions, the lack of direction provided for the students, and the orientation towards the traditional content of the school mathematics curriculum, particularly in the second question above, contrast sharply with more recent examples of investigation in examinations.

Following the Cockcroft Report's recommendations, the new GCSE examinations for all students at 16+ (examined for the first time in Summer 1988) included a teacher assessed component, 'coursework',[5] as well as a traditional externally assessed examination component. The reasons given for the introduction of coursework were two-fold, related

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possible activity is constrained by specifying the number of sums to be made. The task thus fails to meet the criterion of 'openness'.

[4] These questions are certainly not typical of examinations of the time. The syllabus examined by this paper was only taken by students from a very small number of schools.

[5] Until 1991, schools were allowed to choose mathematics syllabuses without coursework. This was a special concession for mathematics (and modern foreign languages) because of the perception that many teachers and students were unfamiliar with the styles of working required. Coursework was compulsory in other subjects from 1988.

both to teaching and to assessment. It was clearly intended that the new examination should serve to encourage the development of the kinds of 'good practice' described by Cockcroft.

> 1.2.1 Coursework should encourage good practice. The elements of such, as defined in Cockcroft paragraph 243, should be in evidence whilst students are undertaking coursework tasks. (ULEAC, 1993, p.3)

In particular, the new form of examination was intended to facilitate assessment of those objectives that could not be assessed effectively by more traditional forms of examination. Thus coursework was to be the means of assessing the GCSE assessment objective 3.17, which stated that candidates should:

> carry out practical and investigational work, and undertake extended pieces of work (DES, 1985, p.3)

By focusing on the style and length of the work undertaken rather than on its mathematical content, this has led to an explicit and innovative valuing of 'processes'. Indeed, it is now commonly perceived that the main purpose of investigational work is to develop and (in the assessment context) demonstrate use of mathematical processes rather than to discover, develop or deploy mathematical knowledge or skills. This focus on process is embodied in the assessment criteria for coursework. For example, the London examining board description of the performance of a candidate gaining a grade A[6] may be seen in Figure 1.

In this list of processes there is no indication of the nature of the mathematical content knowledge and skills to be used and there is thus an assumption not only that it is possible to define and identify achievement such as "chooses efficient strategies" within the context of any type of mathematical problem, but also that the level of difficulty of such achievement will be the same whatever the particular content of the problem. Wiliam (1994) has clearly demonstrated the nonsense of such an assumption in the case of the criteria "formulates general rules" and "makes use of symbols when generalising", showing the substantial differences in complexity in the algebraic expressions arising from two

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[6] The available grades range from A (highest) to G (lowest).

Shows excellent, clear understanding of the task.
Where appropriate extends the task and/or creates sub-problems.
Applies clear reasoning to plan strategies.
Chooses efficient strategies.
Uses appropriate concepts and methods and develops the methods as the work proceeds.
Orders the information systematically and controls the variables.
Uses efficient methods to simplify the task.
Processes data very accurately.
Discriminates between necessary and redundant information.
Plans and schedules a range of relevant mathematical tasks.
Applies a variety of skills, knowledge and procedures to a task.
Recognises patterns.
Makes and tests conjectures.
Formulates general rules.
Where appropriate, makes use of symbols when generalising.
States results achieved and draws and states valid conclusions.
Communicates clearly the work undertaken giving reasons for the strategies used and explaining some assumptions made.
Selects the most appropriate methods for communicating results.
Makes effective use of a range of mathematical language and notation, diagrams, charts and, where appropriate, computer output.
The response to questions is clear, audible, and concise.
Uses and responds to mathematical language relevant to the task and the examination level.
Can explain steps in reasoning in a logical manner, including any assumptions made.
Comments effectively on arguments put.
Responds confidently in a variety of situations, initiates discussion, may ask further questions and sustains conversation.

Figure 1. Grade Description for grade A in Coursework (LEAG, 1989, pp.62)

slightly different interpretations of the same starting point 'Investigate integer-sided triangles':

> the number of integer-sided triangles that can be made with longest side n is given by $\frac{n(n+2)}{4}$ for even n, and $\frac{(n+1)^2}{4}$ for odd n, while the number of such triangles with perimeter n is $\frac{n^2+6n-1+6(-1)^{\frac{n+1}{2}}}{48}$ for odd n not

> divisible by 3, and $\frac{n^2+6n+15+6(-1)^{\frac{n-3}{6}}}{48}$ for odd n divisible by 3, with the results for even n the same as for (odd) $n-3$. (p.52)

The possibility of an individual student being able to demonstrate the processes of generalisation and symbolisation must vary correspondingly between the two interpretations of the problem. I would suggest that there are likely to be similar discrepancies in the levels of difficulty of other processes across different problems, contexts and mathematical content areas. For example, working systematically and controlling variables must surely vary in difficulty according not only to the number of variables involved and the types of relationships between them but also according to the type and familiarity of the context.

Inevitably, in a 'high-stakes'[7] assessment situation such as GCSE, there is considerable concern with the 'fairness' of the examination system. The discrepancies identified above in the difficulty of the 'same' processes within different tasks and within various tasks arising from different interpretations of the same starting point mean that the tasks suggested for students and the variety of student responses allowed or encouraged have to a large extent been restricted to ensure that students only tackle those tasks that will allow them to demonstrate achievement in relation to the criteria rather than enabling them to experience ownership of the tasks, openness and creativity. At the same time, there is concern about the 'fairness' of the grading system and hence with the reliability of teacher assessment of coursework. As Wiliam (1994) reports, there is anecdotal evidence of the success of the publication of assessment criteria and the various training and moderation procedures in achieving a substantial level of agreement in teacher assessments. This reliability has been achieved, however, at some cost to the nature of the object being assessed.

The stereotyping of investigations

While the grade descriptions for GCSE coursework are not intended to be either exclusive or restrictive, it is clear that a substantial number of them may most easily be applied to work within a particular genre that has been labelled as 'Data-Pattern-Generalisation' (DPG) (Wells, 1993) or, rather more derogatorily, 'Train Spotting' (Hewitt, 1992). This genre involves generating numerical data from several examples arising from the given starting point, 'spotting' a pattern in this numerical data and forming a generalised description of the pattern, preferably using algebraic symbols to express the relationship between the variables. The following extract from the descriptors (listed more fully in Figure 1) characterises such an inductive approach but is less applicable to other problem solving approaches.

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[7] GCSE results are widely used to control access to further educational opportunities and to employment.

> Orders the information systematically and controls the variables.
> Recognises patterns.
> Makes and tests conjectures.
> Formulates general rules.
> Where appropriate, makes use of symbols when generalising.
> States results achieved and draws and states valid conclusions.

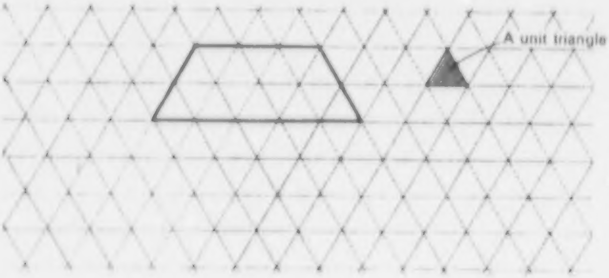
As teachers are concerned to help their students to achieve as highly as possible, it is thus hardly surprising that many of the investigative tasks set provide a structure to guide students through a DPG approach and that, where no explicit structure is provided, students are taught to 'investigate' within this genre, following a standard algorithm:

- generate numerical data from several specific examples
- put the data in a table
- look for a pattern in the numbers in the table
- write down a formula describing this pattern
- check the formula with another piece of data
- explain/prove why the formula works (for the highest attainers only)

The investigative task 'Inner Triangles' (see Figure 2) set by the London examining board (LEAG, 1991) is an example of a starting point that encourages this DPG approach. Indeed, it would be hard to comply with the instructions in the final question "Investigate the relationship between the dimensions of a trapezium and the number of unit triangles it contains" in any other way. It is clearly expected by the examining board and by teachers that all students will start by trying a number of specific examples. Moreover, the 'performance indicators' for this task issued by the examining board suggest that the work of a student gaining a grade D will show "sensible tabulation of results", various levels of sophistication in formulating a generalisation will gain a grade C or B, while only the grade A student is expected to supply "explanation of why the generalised result is as it is" (LEAG, 1991, pp.79-80).

INNER TRIANGLES

The diagram below shows a trapezium drawn on triangular lattice or isometric paper.



The trapezium contains 16 of the unit triangles.
The dimensions of this trapezium are

top length 3 units,
bottom length 5 units,
slant length 2 units?

1. How many unit triangles are there in a trapezium with dimensions
   (a) top length 2 units,
   bottom length 4 units,
   slant length 2 units?
   (b) top length 4 units,
   bottom length 7 units,
   slant length 3 units?

2. Give the dimensions of a trapezium containing
   (a) 8 unit triangles,
   (b) 32 unit triangles.

3. Investigate the relationship between the dimensions of a trapezium and the number of unit triangles it contains.

   In your report you should

   show all your working,
   explain your strategies,
   make use of specific cases,
   generalise your results,
   prove or explain any generalisations.

OPTIONAL EXTENSION

Extend this investigation in any way you wish.

For the extension, the only constraints placed on you are that figures must be drawn on isometric paper and that you must look at figures within figures.

Figure 2. The task 'INNER TRIANGLES' (LEAG, 1991)

All the examples of student work on this problem that I have seen have conformed to the DPG genre, making more or less progress through the algorithm described above. It is possible, however, to imagine an entirely different approach to the problem of relating the dimensions of a trapezium to the number of triangles inside it, for example by using a generic example. The diagram in Figure 3 may be seen as such a generic example and might give rise to (and serve towards an explanation for) a generalisation of the form 'number of unit triangles is the slant height multiplied by the sum of the top and bottom lengths'.

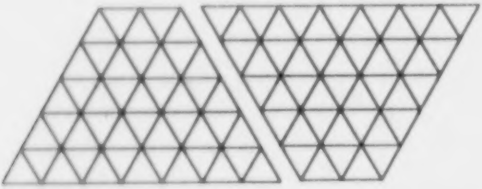


Figure 3. A solution to 'Inner Triangles' by generic example

While it might be possible to match student's work taking such an approach against the general grade descriptors (Figure 1), it does not conform to the expectations of the teachers and examiners. It would thus be likely to cause difficulty for teachers attempting to achieve a 'fair' assessment. The following extract from an interview with an experienced teacher-assessor discussing students' work on the 'Inner Triangles' problem illustrates the teacher's difficulty and suggests the pressure there must be on both teachers and students to conform to the DPG algorithm.

> D With a bright group, I can actually remember a boy doing this and, within twenty minutes with a bright group, giving me the formula. And he'd worked it out very quickly and so - it actually makes it quite a ponderous activity really, cos he'd seen it, he'd got the formula that they're asking for in the generalisation, he knew why it worked, he could explain it, and that was all in one at the beginning of the first lesson.
>
> I So where does he go from there?
>
> D But he still has to fulfill the criteria on a London piece of work. I mean he was a very amenable lad and he did it but it must have been a bit tedious to say the least.

While it is by no means certain that all teachers would respond in such a way, insisting that the student should go through the DPG motions even after successfully solving the problem, it is nevertheless clear that the original idea that investigation should allow students to behave like 'real mathematicians' has fallen by the wayside. Because it is carried out within the traditional context of examination, the student's activity must still aim to do as well as possible in the examination by fulfilling the examiner's expectations (albeit a new set of investigation expectations) rather than to do 'real' or creative mathematics.

In addition to the limiting expectations about students' work created by the DPG stereotype, some further generic conventions have become established which may also be seen to be in conflict with other mathematical values. For example, a convention appears to have arisen that any generalisation ought to be written in words before being expressed symbolically (Morgan, 1994). Again, those students who have a strong capability to think algebraically may well be penalised in the coursework examination and may find their mathematical progress hampered by the need to fufil such requirements. Similarly, fulfilment of the process criterion 'work systematically' has become strongly associated with the production of a one-dimensional table in which the control of variables can be demonstrated. Some teacher-assessors have such clear expectations about the form in which investigative processes may be communicated that a two-dimensional table, however mathematically appropriate to the solution of a two-variable problem, will be interpreted as a failure to fulfil the 'systematic' criterion (Morgan, 1996).

I have described what I consider to be the rather depressing state of the position of investigation within the UK school curriculum. The mathematical and student-centred ideals that originally motivated the advocates of open-ended investigation are still achieved in some classrooms - just as they always have been by energetic, imaginative teachers, confident in their own skills and professionalism. Official endorsement and the introduction of universal assessment have not, however, succeeded in spreading such 'good practice' more widely. Indeed, some of those teachers who at one time enthusiastically embraced the idea of investigation now feel constrained to 'train' their students to carry out the DPG algorithm and argue that there is no time within the curriculum to allow them to work in more open and creative ways. The distortions in the nature of investigation that arise, I have claimed, largely from the requirements of the coursework examination system must draw into question the principle of introducing new assessment practices in order to lead curriculum development. There are especially strong tensions and contradictions between curriculum objectives that incorporate ideals of openness, creativity and student empowerment and the requirements of high-stakes examinations for standardisation and reliability. It seems likely that efforts to increase teachers' mathematical proficiency and professional autonomy and the amounts of time and energy they have available to reflect on and experiment with their own and their colleagues' practices may be a more effective means of effecting such curriculum reforms than the imposition of official curriculum and assessment requirements.

References

ATM. (1988). *Points of Departure*. (Vol. 1). Derby: The Association of Teachers of Mathematics.

Burkhardt, H. (1988). National testing - liability or asset. *Mathematics Teaching, 122,* 33-35.

Cockcroft, W. H. (1982). *Mathematics Counts*. London: HMSO.

DES. (1985). *GCSE: The National Criteria - Mathematics*. London: HMSO.

Ernest, P. (1991). *The philosophy of mathematics education*. London: Falmer Press.

Fielker, D. (1982). Editorial. *Mathematics Teaching, 99,* 2-3.

Hewitt, D. (1992). Train spotters' paradise. *Mathematics Teaching, 140,* 6-8.

Hunt, S., Huyton, P., & Taylor, C. (1988). *Mathematical Starting Points*. Brighton: Spartacus Educational.

LEAG. (1989). *GCSE Syllabuses Mathematics A and B 1991*. London: London East Anglian Group.

LEAG. (1991). *Mathematics coursework tasks and performance indicators (1988-1991)*. London: London East Anglian Group.

Lerman, S. (1989). Investigations: Where to now? In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 73-80). London: Falmer Press.

Morgan, C. (1994). Teachers assessing investigations: The role of algebra, *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 295-302). Lisbon.

Morgan, C. (1995). *An analysis of the discourse of written reports of investigative work in GCSE mathematics*. Unpublished PhD dissertation, Institute of Education, University of London.

Morgan, C. (1996). *Generic expectations and teacher assessment*. Paper presented at the Research into Social Perspectives of Mathematics Education, Institute of Education, University of London.

Ridgway, J., & Schoenfeld, A. (1994). *Balanced assessment: Designing assessment schemes to promote desirable change in mathematics education*. Paper presented at the EARI Email Conference on Assessment.

ULEAC. (1993). *Coursework assessment 1993: Mathematics A*. London: University of London Examinations and Assessment Council.

Wells, D. (1993). *Problem solving and investigations.* (3rd (enlarged) ed.). Bristol: Rain Press.

Wiliam, D. (1994). Assessing authentic tasks: Alternatives to mark-schemes. *Nordic Studies in Mathematics Education*, 2(1), 48-68.

Communication and Negotiation through Open-Approach Method

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Theoretical Background

New demands can be found in Christiansen and Walter (1986), which necessitate changes in the Teacher's role and moves:

1. changes in the distribution of emphasis on the different types of activity.

2. changes in the types of teacher's moves and in the sequence of these in the teaching process.

3. changes in the ways in which the teacher serves as a mediator of mathematical meaning.

We will cite an example of the problem-solving activities between a teacher and learners (Fig. 1). Some of the roles of the teacher at different stages of the teaching and learning process are: as an instructor in teaching mathematical knowledges and skills (Top-Down); as an educator in helping students in problem-solving (Bottom-Up); and as a decision maker in judging whether teaching goes forward or not. The teacher's explication of such roles is integrated with his specific actions and serves in establishing his background and context for the interactions between students' actual and inner activities in connection with their subjective words.

Communication using 'problem-solving' as an organizing principle in Japanese mathematics learning calls for meta-learning under the teacher's support. This communication views mathematics classroom teaching as controlling the organization and dynamics of the classroom for the purposes of sharing and developing mathematical thinking.

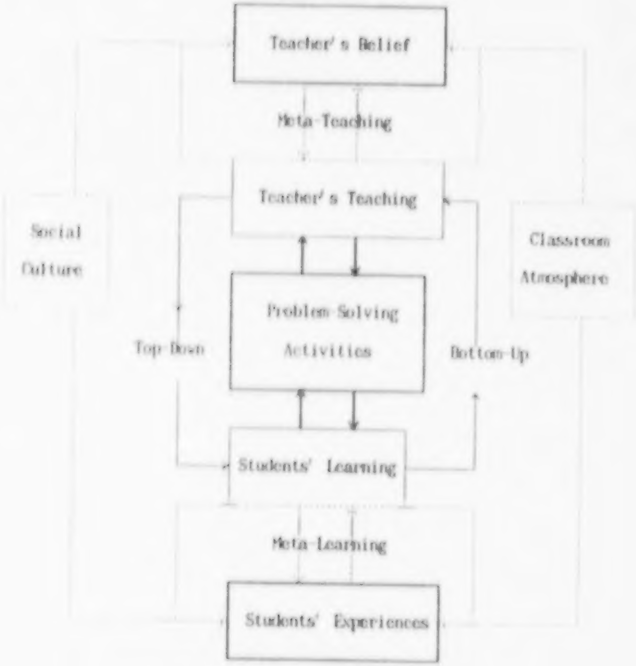

Fig. 1 Problem-Solving Activities

This approach has almost the same features which Bishop and Goffree have already proposed (1986):

(1) it emphasizes the dynamic and interactive teaching;

(2) it recognizes the 'shared' idea of knowing and knowledge, reflecting the importance of both content and context;

(3) it takes into account the student's existing knowledge, abilities and feeling, emphasizing a developmental rather than a learning theoretical approach;

(4) it emphasizes developing mathematical meaning as the general aim of mathematics teaching, including both cognitive and affective goals;

(5) it recognizes the existence of many methods and classroom organizations, and so on.

The educational goal we are concerned with here is that of sharing and developing mathematical meaning. On the basis of this approach we now propose three key concepts which we shall use to structure our ideas and methods of teaching in the mathematics classroom. They are as follows:

Activity---Chosen to emphazise the learner's involvement with problem-solving mathematically, at least, by means of the teacher's presentation of mathematical problems;

Communication---Chosen to accentuate shared meanings with goals and methods of problem-solving underlying teacher's teaching;

Negotiation---Chosen to emphasize the goal-directed interaction of forwards and backwards of the processes in problem-solving, whereby a teacher and learners seek to attain their respective mathematical goals.

We shall analyze the problem-solving processes of students in the mathematics classroom based on 'Open-ended' problems using the three key concepts mentioned above.

Mathematical Problem Solving by 'Open Approach' Method

We hope to become more aware of an information-oriented society which consists of the communications between the teacher's belief and the students' experiements about problem-solving activities, that is, the students's speaking and writing abilities about some key words which the teacher says. We use non-routine problems as problem situations (Christiansen & Walter, 1986). The problems we use here for our research, are non-routine problems which can be solved by students independently in the mathematics classroom (Nohda, 1983, 1986).

The problems we use here for our research, have a feature of pattern-finding. One of the dominant themes of cognitive research into problem-solving in recent years has been pattern finding. They are able to solve the problem when they find a suitable 'pattern' in the problem. Whereas they have some feelings of difficulty in solving their problems when they do not find a suitable 'pattern'. To study their mathematical activities by means of the strategies and difficulties of problem-solving is to make clear how they find more suitable patterns through some interaction between the teacher and students, and among students (Silver, 1979).

Research Question

For the purpose of this study, we consider the mathematical activities in the following two cases. One case is the underlying pattern in the problem, that is, the nature of the problem itself. The other is the feature of the strategies in students' problem-solving. The former means the structure of problem and the rule in it, etc.. The latter is the mode of action applied in students' problem-solving. Therefore, in order that students might do better in their problem-solving, it is necessary for them to share the understandings of problems through some communications with their teachers. For students who fail to understand the problem or feel difficulty in solving it, the reason would be that there is no sharing of understanding or way of solving the task through the interactions between tasks and students under the teacher's instruction.

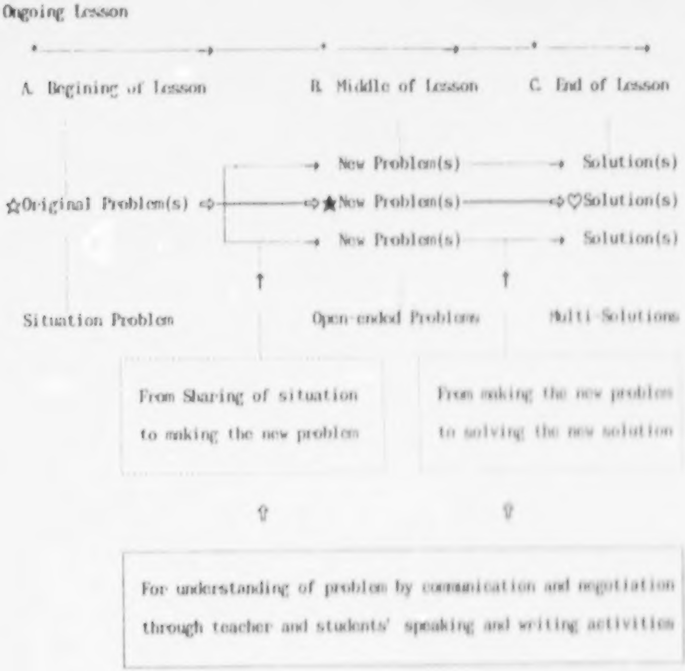
Ongoing Lesson
A. Begining of Lesson
B. Middle of Lesson
C. End of Lesson
☆Original Problem(s)
New Problem(s)
★New Problem(s)
New Problem(s)
Solution(s)
♡Solution(s)
Solution(s)
Situation Problem
Open-ended Problems
Multi-Solutions
From Sharing of situation to making the new problem
From making the new problem to solving the new solution
For understanding of problem by communication and negotiation through teacher and students' speaking and writing activities

Fig. 2 Communication and Negotiation through Open-Approach Method

Empirical Results

For Example: Original problem is a telephone lines' problem

Telephone lines' Problem

There are some houses. Now, we connect a house to another by a direct telephone line. We put just one telephone line between each other.

Now make up your own problems using the situation above and write them down. Make problems as many as you can.

If you can solve your problems, you may write your own answers.

Example 1 (Grade 5)

A teacher wrote the problem on the blackboard.

The teacher asked students to make their own problem using this situation.

A few minutes later, one student posed a question.

Student A : "I can't understand how to connect houses together."

Teacher : "Would you consider your problem in the case of three houses?"

Student A : Student A drew Fig. 3. And She considered her teacher's suggestion by moving her finger on the figure. (Fig. 3)

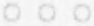

Fig. 3 : Student A drew on the blackboard.

Student B : "We have to connect every set of two houses with one telephone line. So you should connect these two houses like this."

Student B added a line on Fig. 3. (Fig. 4)

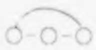

Fig. 4 :Student B added a line.

Student C : "This figure looks strange, doesn't it ?

So I move this house below those two houses.

The previous figure (Fig. 4) changes into the form of a triangle."

Student C moved the right end mark "o" below the other two houses.

And drew two lines. (Fig. 5)

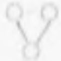

Fig. 5 : Student C drew two lines.

Other students : "Well, well, I got it !"

(Some students praised his idea.)

Teacher : "You got it ? This figure says how to connect every set of two houses with one telephone line !"

After these conversations, one student made the same problem as Problem 2 mentioned previously.

Every student tried to solve this problem.

Problem 2. How many telephone lines are there for 20 houses ?

Student A's Solution : $1 + 2 + 3 + \ldots\ldots + 17 + 18 + 19 = 190$

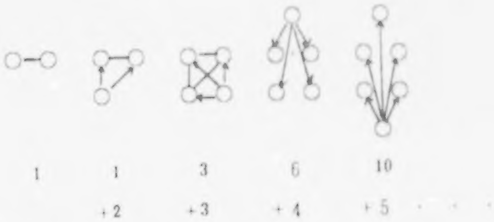

Fig. 6 : The figure Student A drew on her note.

Student A deduced an additional method using Student C's message. In this case, Student C's addition of two lines influenced Student A's solution.

Student B's Solution : $(20 - 1) \times 20 \div 2 = 190$

Student C's Solution : $1 = 2 \times 0.5$ (for 2 houses) , $3 = 3 \times 1$ (for 3), $6 = 4 \times 1.5$ (for 4), $10 = 5 \times 2.0$ (for 5), $15 = 6 \times 2.5$ (for 6), $\ldots 20 \times 9.5 = 190$ (for 20)

Student C drew 5 pictures which illustrated the cases from 2 houses to 6 houses. And he deduced one formula such as "The number of telephone lines $= N \times 0.5(N-1)$; N = The number of houses", from the relation between the number of houses and lines. Finally, he was able to solved this problem, but he could not tell us why his method was effective.

Educational Implications

One of the reasons why we picked this example is that it illustrates some characteristic students' activities such as "posing a new question", "discussing and sharing their ideas", "working together (collaboration)", and "summarizing their discoveries in writing".

And the other reason is that this case shows how a receiver is influenced by a message. When solving Problem 2, Student A solved the problem in a better way as a result of communication was not a directed result of what Student C intended to communicate, but rather that what Student C said and drew helped Student A to better understand the problem and move toward a solution.

The educational implications from our study of actual communication in the context of problem-solving by using the open-ended problem in the mathematics classroom consist of the following:

1. In the study of students' strategies and difficulties in problem-solving, we should concentrate on both the pattern of the problem and the mode of the students' acts of problem-solving by themselves. We suggest here that students need to act initially by themselves to solve the problem and then, through communication establish adequate patterns in the modification of their initial acts.

2. Some excellent students can solve the problem by finding the essential mathematical patterns underlying in the problem. The teacher should support these students to promote their more advance solution after using the teacher's primitive method. They are urged independently to find the advanced solutions. Thus, they will become better problem solvers in the future. These are the aspects of communication and negotiation which are most important for the teacher. They find the new ideas of relation between an additional method and a multiplicational method.

3. Many normal students cannot solve the problem at hand. In such situations, the teacher has to advise them to recall some familiar problems which they have solved. Being able to find a familiar problem will help them solve the problem in the near future. These are aspects of communication and negotiation, too.

References:

Bishop, A. J. (1985), The Social Construction of Meaning-a Significant Development for Mathematics Education ?. For the Learning of Mathematics, 5(1), 24-28

Bishop, A. J. & Goffree, F. (1986), 'Classroom Organization and Dynamics', in B. Christiansen, A. G. Howson and M. Otto(eds) Perspective on Mathematics Education, Reidel, Dordrecht.

Desforges, C. (1989), Mathematics Teaching The State of the Art", Ernest, P., The Falmer Press.

Elliott, P. C. & Kenney, M. J. (Eds), 1996 Yearbook: Communication in Mathematics, K-12, and Beyond, Virginia, NCTM INC.

Emori, H. & Nohda, N. (1992), Communication Process in Learning Mathematics--Communication Gap and Filter Effect--, Proceedings of the Sixteenth PME Conference, Vol. 3, 159, Durham, USA

Emori, H. (1993), The Mechanism of Communication in Learning Mathematics, Proceedings of the Seventeenth PME Conference, Vol. II, 230-237, Tsukuba, Japan.

Kumagai, K. (1987), 'A study on "Sharing Process" in the Mathematics Classroom', Annual Report of Graduate Studies in Education, Vol 11.

Nohda, N. (1983), "A Study of 'Open-Approach' Strategy in School Mathematics Teaching", Touyoukan Publishing Co., Doctoral Dissertation, (in Japanese). This abstract is 'The Heart of 'Open-Approach' in mathematics Teaching', The Proceedings of ICME-JSME Regional Conference on Mathematical Education, 1983.

Nohda, N. (1986), "Open Mind of Children in Arithmetic Teaching", Koubundou Publishing Co., (in Japanese).

Romberg, T. A., (1998), "Can Teachers be Professionals?", "Effective Mathematics Teaching", Research Agender for mathematics Education, NCTM.

Silver, E. A., (1979), "Student perceptions of relatedness among mathematical verbal problems', N.C.T.M., J.R.M.E. Vol.10, No.3,

Silver, E. A., (1987), "Problem solving, Tips for Teavhers'. NCTM Arithmetic Teacher, vol. 34. No.5

Shimada, S., (1977), "Open-Ended Approach in Arithmetic and Mathematics—A New Plan for Improvement of Lessons", Mizuumi-Shobou, (in Japanese)

Sowder, L., (1985), "Cognitive Psychology and Mathematical Problem Solving: A Discussion of Mayer's Paper', Silver, E. A. (Ed.), "Teaching and Learning Mathematical Problem-Solving: Multiple Research Perspective", Lawrence Erlbaum Associates.

von Glasersfeld, E. (1987), "Learning as a Constructive Activity': Janvier C. (Ed.), "Problems of Representation in the Teaching and Learning of Mathematics", Lawrence Erlbaum Associates.

Voight, J. (1996), Negotiation of Mathematical Meaning in Classroom Process: Social Interaction and Learning Mathematics. In L.P.Steffe, P.Nesher, P.Cobb(Eds.), Theories of Mathematical Learning(pp.21-50), New Jersey: Lawrence Erlbaum Associates, Publishers.

Use of problem fields as a method for educational change

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The task of the comprehensive school is to train pupils to cope with as many of the many-sided problems in their future life as possible. Therefore, the teaching of mathematics should be developed so that the knowledge and skills learnt are useful for most pupils. Up to the present time, mathematics has been taught as end in itself, i.e. mathematics for mathematics' sake. (Martin 1986, 13) However, some changes are now coming apparent in the conceptions of how mathematics should be taught.

A constructivist's view of learning

During the 1980's, the theory of constructivism which was developed in a cognitive psychology has increased its scope to an explaining theory of learning (e.g. Schoenfeld 1987; Davis & al. 1990; Ahtee & Pehkonen 1994): As its starting point, there is the conception that when a learner acquires new knowledge he is at least partly constructing it actively from scratch. The learner is not only adding the new information to his knowledge store, but in the case of understanding what he is learning he is constructing (and he ought to construct) connections between the new knowledge and his old stored knowledge, i.e. he at least partly constructs his knowledge structure. This process of constructing new connections is essential for learning. Therefore, learning should be understood as a process - i.e. as a sequence of mental actions. New knowledge being assimilated in an interpreted form in the pupil's existing knowledge structure, which will as a result change it.

Communication in mathematics learning. Traditionally, people have thought that language and mathematics have no connections and that mathematics creates its own language for its needs. However, the 1980's have seen an increase in interest in the meaning of language and communication for mathematics teaching and learning. This interest is apparent in many other opinions concerning mathematics teaching and learning (e.g. Cockcroft 1982, Anon. 1989, NCSM 1989, NCTM 1989). Thus one of the central themes in the Cockcroft-report is language and communication. In the report, they demand among other things that emphasis should be placed on "discussion between teacher and pupils and between pupils themselves" in mathematics teaching for pupils of all ages (Cockcroft 1982, 71).

Constructivism emphases the learner's contribution in the learning process. If the teacher pays attention to this, he is compelled to listen to his pupils and to follow their thinking process as well as to try to understand them. One method of achieving this is to use discussion as an element of teaching. Pupils have very often preconceptions (or misconceptions) about the concept to be learned. The teacher can understand the pupils' way of thinking e.g. through listening to discussions between pupils. He can then use these views as a starting point for his teaching. (Schoenfeld 1987, 29)

Activity in mathematics learning. When teaching mathematics in the comprehensive school, the pupil's mathematical activity should form an essential part of teaching (cf. Cockcroft 1982, Spiegel 1982, Walsch 1985, Anon. 1989, NCTM 1989). In the Cockcroft-report, it is stated that there is not only one definitive style for the teaching of mathematics but "approaches to the teaching of a particular piece of mathematics need to be related to the topic itself and to the abilities and experiences of both teachers and pupils". Nevertheless, some elements should be present in mathematics teaching to pupils of all ages, e.g. appropriate practical work. (Cockcroft 1982, 71) In the NCTM-Standards, the role of the pupils is described through the verbs: investigate, explore, describe, develop, use, apply, invent, relate, model, explain, represent, validate (e.g. Lindqvist 1989).

The constructive view of learning stresses a learner's contribution to the learning process. Qualitatively successful learning, so-called deep learning can only happen when the learner himself is actively working with new subject matter. This leads to an emphasising of the pupil's own activity, i.e. in the learning situation, there should be opportunities for the pupils to originate activities.

A conception of mathematics. The following is a short description of a view of mathematics whose roots lie among other things in the books of Lakatos 1977 and Davis & Hersh 1985. This explication is due to Zimmermann (1991), but e.g. the National Council of Teachers of Mathematics has also expressed a similar view (Lindqvist 1989, 1-4):

- Mathematical content is more important than mathematical formalism.
- Mathematical thinking processes are at least as important as mathematical results.
- Good mathematics is not characterised by the lack of mistakes, but the quality of ideas.
- When emphasising the level of mathematical rigour one ought to take into account the pupils' level of mental development.
- The playful and aesthetic aspects of mathematics should also be stressed.
- When doing mathematics, it is important to have an open mind as other subjects and applications are concerned.
- Mathematics ought to be understood as a net-work-system which is itself part of the broader net-work-system concerned with the rest of life. (Therefore, flexible and net-like thinking should be encouraged.)
- The role of logic in mathematics is the same as the function of grammar in writing.
- The meaning of mathematical language is more important than its symbolic expressions.

What is a problem field?

In order to implement the changes described above, it is generally suggested that the teaching of school mathematics should move towards a more open direction. This can happen through using so-called open tasks in mathematics classrooms.

As our starting point, we are going take the definition of a problem which is commonly used in the literature (e.g. Kantowski 1980, 195): We will define a *problem* as a situation where individuals are compelled to connect known information in (for them) new ways, in order to accomplish a task. If they can immediately recognise the actions needed to do the task, then it will be a standard (routine) task.

Often we will consider a sequence of problems which are connected to each other. Such a set of connected problems will be called a *problem field*. From any problem one can generate a problem field through changing those conditions given in the task (see Pehkonen 1986, 1992).

Solvang talks about problem domain and its generating problem meaning the same (Solvang 1994).

Two examples of problem fields will be described below: Polygons with matchsticks, and Number Triangle. They represent those problems I have developed to be used in heterogeneous classes in Finnish comprehensive schools. In each problem field, the difficulty of the problems ranges from very simple ones that can be solved by the whole class, to harder problems which only the more advanced students might be able to solve.

One of the characteristics of problem fields is that they are not bound to a fixed class-stage, but are suitable for mathematics teaching from primary level to teacher in-service education. The role of the easier problems in problem fields is especially to reinforce the problem solving persistence of pupils. The most important aspect of all in these problems is the way in which they are introduced to a class: The problem field ought to be given gradually to pupils, and the continuation should be related to the pupils' solutions. Instead of the answers and results which are not given here, the process of problem solving is of paramount importance. The most important aspect is the use of pupils' own creative power. The level to which the teacher takes the problem field, depends on the pupils' answers.

Polygons with matchsticks

Twelve matchsticks (or cocktail-sticks, etc.) will be needed to concretisize the problems. The starting situation is the following:

> *With twelve matchsticks one can make a square (Fig. 1) the area of which is 9 au (au = area units).*

From this situation has been developed a sequence of problems (a problem field). Firstly, we will choose another area, but have the perimeter of the polygon constant. Thus, in each problem the perimeter of the polygon should be made up of 12 matches

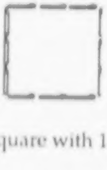

Fig. 1. A square with 12 matches.

- *Can you use twelve matches to make a polygon with an area 5 au?*

If we are willing to give more thinking time to slower pupils, the faster ones can be asked to find another (perhaps also a third) solution. Fig. 2 shows some of the pupils' solutions.

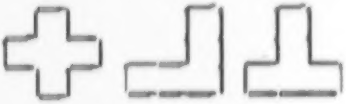

Fig. 2. Some polygons with the area 5 au.

As the next question, we might ponder the number of different solutions.

- *How many different polygons of 5 au can you make with twelve matches? Can there be more than ten different solutions?*

The pupils will probably find many of the solutions. But there are still more complicated solutions which they probably will not find. The following stage might be the comparison of the different solutions found by pupils. How many of all different solutions can be found when the whole class is working together?

Another easier direction to vary the problem is to change again the area.

- *Is it possible to use twelve matches to make a 6 au (or 7, 8 au) polygon?*

The solutions in Fig. 3 can be found easily. But are there any other solutions in each case? And how many different ones?

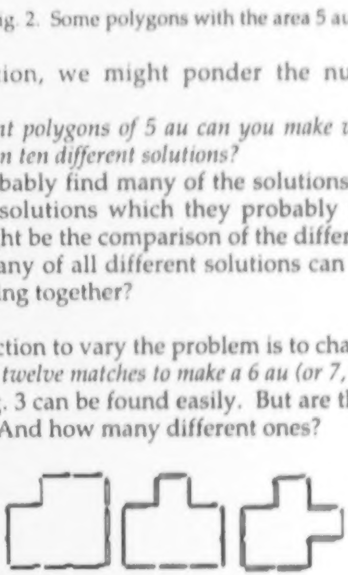

Fig. 3. Polygons with the area 8, 7, 6 au.

The method of cutting out a corner from a rectangular polygon, as in the Fig. 2 and 3, is successful until the area 5. But the question of smaller areas is more complicated, since we are compelled to change the method.

- *Is it possible to use twelve matches to make a 1 au (or 2, 3, 4 au) polygon?*

With the aid of the Pythagorian theorem, one can construct polygons with an area of 4 au and 3 au. It should also be possible to find the general solution: the parallelogram. But the question of different solutions and their number in the case of area 2 au (or 1 au) is according to my experience really hard one.

Still one enlargement of the problem field is to ask for greater polygons than area 9 au:

- *Using twelve matches is it possible to make a polygon whose area is greater than 9 au?*

This seems to be a hard one, since also in teacher pre-service and in-service courses, this question has so far not been solved.

Teaching experiences. Within the last ten years, I has worked through the problem field "Polygons with matchsticks" with many groups of teachers both on pre-service and in-service training courses as well as with some school classes in the lower secondary school.

Usually, the problem field has taken about 30 minutes. In the teacher groups, we have found many polygons with areas of 4, 5, 6, 7, 8 au. But those with areas smaller than 4 au seem to be very complicated to construct. Only in a couple of groups has somebody produced the general solution: a parallelogram. When all new ideas for solutions from the group have tried up, I have set the last problem worked on by the group as homework. My main reason for introducing problem fields in teacher training was to describe to teachers how to deal with a problem field and to give them an idea how pupils feel when solving them. The teachers have usually liked the problems and the way we have dealt with them.

In the autumn 1996, I had an opportunity in Jena (Germany) to work with a group of highly talented pupils from the local upper secondary school. Among others, we worked through this problem field, and the pupils found the problems with smaller areas (smaller than 5 au) very challenging. Also for them, it took some time to find out other solutions than the parallelogram.

Number Triangle

The starting situation for this problem field is given in Fig. 4:

> *A triangle where the corner are free and some numbers are fixed on the sides.*

The first problem is, as follows:

- *What numbers should be placed in the blank circles in the triangle in Fig. 4 so that the sum of the three numbers on each side is equal?*

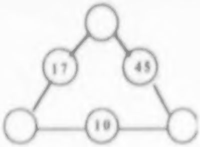

Fig. 4. Number Triangle problem.

Since there are many solutions for this problem, we will usually ask as next:

- *Can you find another solution?*

and

- *How many different solutions could there be?*

There are infinite number of solutions, but pupils usually do have difficulties in finding more than one.

Another direction might be to enlargen the number domain accessable. As a rule, pupils suppose that the numbers placed into empty circles are natural numbers.

- *Is it possible to use negative numbers in the circles?*

At the first glance, pupils in the lower secondary school may answer this question in the negative, but they usually find after working through some examples (e.g. as a home work) that it is possible.

By fixing the side sum, one gets a different kind of problem. If you want your pupils to practice with negative numbers, you may put zero (or some negative number) for the sum.

- *Can you find a solution where the triangle's side sum (i.e. the sum of the numbers on the same side) will be 80?*

In the lower secondary school, pupils usually solve this kind of problem by trial-and-error method. If they don't find out any general rule, this might be a good place to practice systematic trial-and-error.

An interesting enlargenment is to ask for the possible number domain:

- *Which numbers are possible as the triangle's side sum?*

Most pupils will suggest here integers. After negotiations, they might see the possibilities of fractions, but irrational numbers seem to be unpossible for them to think and invent.

The usual question of generalisation may be discussed also within this problem field:

- *How could you generalise the problem?*

School experiences. In the fall of 1987, I experimented with the use of Number Triangle as a separate problem in one grade 7 class in Helsinki. My original objective was to improve the pupils' mental calculation skills in an unusual way, as well as their problem solving skills. But the Number Triangle led to an interesting problem sequence (a problem field) which is described above, in the frame of which pupils were training among other things calculations with negative numbers for some weeks. The whole problem field was dealt with in six different lessons, but usually only as a part of the lesson. The entire class was eager to find solutions for separate problems and to search for general solving principles. Some pupils were telling me their new solutions when they met me outside of mathematics class. If we did not do the problem field during some lesson, they asked when we would continue and were willing to show the solutions they had found.

"Open tasks in mathematics" -project

In Helsinki (Finland) and Hamburg (Germany), the research project "Open tasks in mathematics" was started in the fall of 1987 and finished in the spring of 1992 (Pehkonen & Zimmermann 1989, 1990). The project tried to clarify the affect of problem fields on pupils' motivation, the methods and how to use them.

The aim of the project

The purpose of the research project was to develope and foster methods for teaching problem solving in lower secondary schools. We tried to stay within the frame of the "normal" teaching, i.e. in the frame of the valid curriculum, and to take account the teaching style of the teachers when using problem fields.

The objectives of the research project can be categorised into six fields of emphasis: (1) To clarify possibilities and methods for the use of problem fields in teaching. (2) To foster pupils' attitudes against mathematics and mathematics teaching. (3) To develop the flexibility of pupils' thinking. (4) To examine how pupils' problem solving ability develops when normal teaching methods are used. (5) To develop teachers' conceptions of mathematics and mathematics teaching. (6) To clarify whether there are any differences in the teaching of these points in Germany (especially Hamburg) and Finland.

On the realisation of the project

In the pilot study of the project during 1987-89, the research design was tested, measurement instruments were developed and the problem material was worked into its final form. The main experiment was planned to begin in the fall of 1989 both in Hamburg and in Helsinki with ten grade 7 classes and to be continued with those classes through the whole lower secondary school (up to grade 9), i.e. to spring 1992. Unfortunately, the German counterpartner was not able to get the required finance, and therefore the research project was realized only in Helsinki with the financial support the Academy of Finland.

Both in the beginning and at the end of the experimental phase, teachers´ and pupils´ conceptions of mathematics teaching have been gathered using questionnaires and interviews. Because the group of ten teachers is not at all representative, about 50 teachers (comparing group) have been also tested with the same questionnaire.

In the main experiment, experimental groups 1 and 2 differed in the point that from the mathematics lessons of experimental group 1 about 20 % (i.e. once a month about 2-3 lessons) was reserved for dealing with problem fields. There was a questionnaire for each problem field in which the pupils´ conceptions of using that problem field were ascertained. The teachers´ conceptions of using problem fields were obtained with short interviews. The teachers in experimental group 2 were told that they are participating in an experiment, whose aim was to investigate the development of pupils´ problem solving skills in natural teaching environment. They were not told anything about problem fields nor the experimental group 1. Pupils in experimental groups solved in their class work some open problems which were the same for both groups.

On the results of the project

Here we will discuss briefly two main groups of the results: Firstly, results in respect of pupils (for more details see e.g. Pehkonen 1995), and secondly, some results concerning teachers (for more details see e.g. Pehkonen 1993).

On the results in respect of pupils. Summarizing the results of the research project, one could state that the pupils experienced the problem fields used as an interesting form of learning mathematics. They liked most of them very much, and were motivated and activated also during other parts of mathematics lessons. Their mathematical views did not change statistically significantly. But the non-significant changes in questionnaire, classroom observations and the teacher's evaluations indicate that there existed a change, and the change was in most cases positive. Furthermore, when the experiment was finished, one of the teachers (Terry) wanted to continue working with similar materials later on, and she also has done it.

When interpreting the results, there are many critical points to take care of. For example, in measuring pupils' mathematical views, there seem to be some problems. The use of the paired t-test shows a variety of changes in the pupils' responses (the initial–final comparison), also in those statements where no statistically significant difference between the initial and final testing was found in the unpaired t-test. Thus, the pupils answers to the statements seem to change in three years much, but not each time in the same direction. So the question arose, whether the pupils have any fixed views to measure on that age. Is is sensible to try to figure out pupils' mathematical views with a questionnaire?

On the results concerning teachers. Based on the research findings, some questions arose. Firstly: On which reasons do the teacher actually form his assessment of the selection of an open-ended problem (or more generally of mathematical teaching material)? It seems that some of teachers base their assessment on the convenience to use the material. This explanation can be heard by the publishers when they demand from the authors of learning materials that the material should be easy to use. And this demand seems to be justified.

Secondly: Which kind of objectives should we pose for those conducting the change in teaching? In the research project, we aimed with open-ended problems to cause change in mathematics teaching. In the research findings, we see that about one third of the reasons given by the teachers are connected with the convenience to use the material. Thus in order to reach change with the aid of teaching material, one may choose at least between two ways: (1) One emphasizes the pupil-centerness and the mathematical content of the tasks. This leads to the problem of teacher in-service education. (2) One is satisfied with the offering of easy-to-use materials to teachers. This leads to the problem of producing materials.

Endnote

The results suggest that the open-ended approach, when used parallel to the conventional teaching methods, seems to be a promising one. The pupils prefered this kind of mathematics teaching where one important factor was the freedom let to pupils to decided their learning rate. The use of open-ended problems, in the form of problem fields, has been experienced to be a so promising approach that there is now a ready written text book for mathematics teaching of grades 7–9 with this approach (Espo & Rossi 1996). The text book has been accepted by the National Board of Education for a larger survey with help of which they are trying to find out its possibilities at large.

Literature

Ahtee, M. & Pehkonen, E. (eds.) 1994. Constructivist Viewpoints for School Teaching and Learning in Mathematics and Science. University of Helsinki. Department of Teacher Education. Research Report 131.

Anon. 1989. Mathematics in the National Curriculum. Department of Education and Science and the Welsh Office. London: H.M.S.O.

Cockcroft Report 1982. Mathematics counts. Report of the Committee of Inquiry into the Teaching of Mathematics in Schools. H.M.S.O.

Davis, P.J. & Hersh, R. 1985. Erfahrung Mathematik. Boston: Birkhäuser.

Davis, R.B., Maher, C.A. & Noddings, N. (eds.) 1990. Constructivist Views on the Teaching and Learning of Mathematics. JRME Monograph Number 4. Reston (VA): NCTM.

Espo, K. & Rossi, M. 1996. Matka matematiikkaan [A journey to mathematics]. Helsinki: Edita.

Kantowski, M.G. 1980. Some Thoughts on Teaching for Problem-Solving. In: NCTM Yearbook 1980, 195-203. Reston (VA): Council.

Lakatos, I. 1977. Proofs and Refutations. Cambridge: Cambridge University Press.

Lindqvist, M.M. 1989. It's Time to Change. In: NCTM Yearbook 1989, New Directions for Elementary School Mathematics, 1-13. Reston (Va.): Council.

Martin, J.-C. 1986. A Necessary Renewal of Mathematics Education. In: Mathematics for All (eds. P. Damerow, M.E. Dunkley, B.F. Nebres & B. Werry), 13-17. Science and Technology Education. Document Series No. 20. Paris: Unesco.

NCSM 1989. Essential Mathematics for the Twenty-first Century. Math. Teacher 82 (6), 470-474..

NCTM 1989. Curriculum and Evaluation Standards for School Mathematics. Reston (Va.): Council.

Pehkonen, E. 1986. Über die Bedeutung der Knobelaufgaben. In: Beiträge zum Mathematikunterricht 1986, 227-230. Bad Salzdetfurth: Verlag Franzbecker.

Pehkonen, E. 1992. Using Problem Fields as a Method of Change. The Mathematics Educator 3 (1), 3–6.

Pehkonen, E. 1993. On Teachers' Criteria to Assess Mathematical Activities. In: Proceedings of the seventeenth PME conference (eds. I. Hirabayashi, N. Nohda, K.

Shigematsu & F.-L. Lin). Volume I, 220–227. University of Tsukuba, Tsukuba (Japan).

Pehkonen, E. 1995. On pupils' reactions to the use of open-ended problems in mathematics. Nordic Studies in Mathematics Education 3 (4), 43–57.

Pehkonen, E. & Zimmermann, B. 1989. Offene Probleme im Mathematikunterricht. In: Mathematics Education Research in Finland. Yearbook 1987-88 (ed. P. Kupari), 55-77. Institute for Educational Research. Publication series B. Theory into Practice 39. University of Jyväskylä.

Pehkonen, E. & Zimmermann, B. 1990. Probleemakentät matematiikanopetuksessa. Osa 1: Teoreettinen tausta ja tutkimusasetelma. [Problem Fields in Mathematics Teaching. Part 1: Theoretical Background and Research Design.] Helsingin yliopiston opettajankoulutuslaitos. Tutkimuksia 85.

Schoenfeld, A.H. 1987. Cognitive Science and Mathematics Education: An Overview. In: Cognitive Science and Mathematics Education (ed. A.H. Schoenfeld), 1-31. Hillsdale (NJ): Lawrence Erlbaum Associates.

Solvang, R. 1994. Thoughts about the free phase in connection to the use of generator problem. In: Selected Topics from Mathematics Education 3 (ed. R. Solvang), 56–67. Centre for Teacher Education and In-Service Training, University of Oslo.

Spiegel, H. 1982. Zur Situation des Mathematikunterrichts in der Primarstufe - Tendenzen, Probleme, Perspektiven. In: Beiträge zum Mathematikunterricht 1982, 112-121. Hannover: Schroedel Schulbuchverlag.

Walsch, W. 1985. Gedanken zur Realisierung des "Tätigkeitskonzeptes" im Mathematikunterricht. J. Math. Didakt. 6 (1), 3-14.

Zimmermann, B. 1991. Offene Probleme für den Mathematikunterricht und ein Ausblick auf Forschungsfragen. In: International Reviews on Mathematical Education [= ZDM] 23 (2), 38–46.

On the Use of What-If-Not Strategy for Posing Problem

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Introduction

In mathematics education community, mathematical thinking has been considered to be an important aspect of mathematics learning and teaching. However, as Brown and Walter(1990) mentioned, much of mathematical thinking begins with the assumption that we take the "given" for granted. We are trained to begin a proof be first stating and accepting what is given(p.32.). They emphasize that we need a different notion than that of merely specifying and accepting the given as it is used in problem solving. They, then, describe What-If-Not strategy, i.e., the second phase of problem posing as a means to bring about an accomplishment to the aims of mathematics learning and teaching(Brown & Walter, 1969, 1970, 1990). However, What-If-Not strategy has meant many things to many researchers. Thus, in order to confirm that this strategy would be really an effective means to bring about such an accomplishment, this paper attempts to describe various interpretations of What-If-Not strategy and to clarify the aims of this strategy.

Interpretation of What-If-Not strategy

Although researchers often view What-If-Not strategy proposed by Brown & Walter to be an effective method for making problems(Silver, 1993; Zimmermann, 1986), interpretation of the five levels of the strategy varies depending on researchers. For example, Tejima(1992) views What-If-Not strategy as below. First, students have to make clear the attributes of the confronting problem, then they have to question each attribute with "What-If-Not." For him, this assumes that it is the way for generating new problems. The following illustrates how Tejima interpreted Brown and Walter's five levels of problem posing.

Level 0 Choosing a Starting Point:
Selecting the confronting problems or situations
Level 1 Listing Attributes:

Extracting the attributes from various perspectives

Level 2 What-If-Not-ing:
Applying "What if not" question to each selected attributes

Level 3 Question Asking or Problem Posing:
Creating new problems

Level 4 Analyzing the Problem:
Analyzing the created problem and solving that problem

(The italic parts are added by Tejima)

For Tejima, he asserts that new problems are created by changing some parts of the attributes of the given, i.e., a given problem and a given situation. Shioda(1991) interprets Level 0-3 similar to Tejima does. However, he, interprets Level 4 quite differently. He suggests that in order to consider how the changing attributes influence on the solutions, students should compare the given problem with the created problem and their solutions. In sum, Tejima views Analyzing just ended before getting the solution of the created problem but Shioda includes the processes after getting the solutions.

Munehiro(1982) suggests that to solve the created problems in Level 3 is the meaning of Level 4. Through that solving process, a new problem would be emergent. However, Munehiro does not focus on the relationship between the created problem in Level 3 and the emerging problem in Level 4.

As it can be seen that the above-mentioned views of Analyzing in Level 4 might result from the unclear aims of What-If-Not strategy. Thus, next section describes the aim of What-If-Not strategy.

The aims of What-If-Not strategy

Brown & Walter(1990) very much emphasize on to ask a question or to pose a problem rather than to answer a question. They describe a new perspective on viewing questions. They, then, propose two phases of problem posing: accepting and challenging. Particularly, they argue that it is challenging the given which frequently opens up new vistas in the way we think. They wrote: Only after we have looked at something, not as it "is" but as it is turned inside out or upside down, do we see its essence or significance(p.15). They rephrase the term "challenging" with "What-If-Not?". With What-If-Not strategy, can reveal our students with several very important points about mathematical thinking, which is hindered by taken the "given" for granted as mentioned before. Taken the "given" for granted we lead our students to begin a proof. In contrast, Brown and Walter argue that there is certainly much more to mathematics than proving things. They list some reasonable mathematics activities when using What-If-Not strategy as follows:

1) Coming up with a new idea
2) Finding an appropriate image to enable us to hold on to an old one
3) Evaluating the significance of an idea we have already learned
4) Seeing new connections

I propose that the just described activities are really the aims of What-If-Not strategy proposed by Brown and Walter.

In conclusion, based on the assumption the aims of What-If-Not strategy have already been described as such. This would make us(researchers and teachers) clearly how to implement "What-If-Not" strategy as an effective means to bring about an accomplishment to mathematics learning and teaching.

Reference

Brown, S. I. & Walter, M. I. (1969). What if not? Mathematics Teaching, 46, 38-45.

Brown, S. I. & Walter, M. I. (1970). What-if-not?: An elaboration and second illustration. Mathematics Teaching, 51, 9-17.

Brown, S. I. & Walter, M. I. (1990). The Art of Problem Posing (second edition). Philladelphia: The Franklin Institute Press.

Munehiro, Y. (1982). The significance and method of problem posing: Through the application of What-If-not strategy. Tsukuba Journal of Educational Study in Mathematics, 1, 57-68.

Shiota, N. (1991). A study of the problem-solving process emphasizing on problem generating: as a perspective for problem formulating and problem posing. Joetsu Journal of Mathematics Education, 6, 95-104.

Silver, E. A. (1993). On Mathematical Problem Posing. In: Ichiei Hirabayasi et al. (Eds.), Proceeding of the 17th International Conference for the PME, Vol. I. (pp.66-85). Tsukuba, Japan: Univ. of Tsukuba.

Tejima, K. (1992). The strategy of problem posing. Joetsu Journal of Mathematics Education, 7, 17-30.

Zimmermann, B. (1986). From problem solving to problem finding in mathematics instruction. In P. Kupari (Ed.), Mathematics Education Research in Finland (1985Yearbook) (pp.81-104.). Jyväskylä:Institute for Educational Research.

Learning Mathematics Through Exploration of Open-ended Tasks: Describing the Activity of Classroom Participants

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Summary

This study was based on a belief that open-ended tasks are one way of encouraging students to become learners of mathematics by doing and even creating mathematics for themselves. It aimed to examine the effect of a class program based mainly on open-ended content specific tasks.

The program was planned with the teacher to ensure that the questions were suitable for the class. It seemed that the program was delivered in a way which was compatible with the intentions of the study. Most questions asked were open-ended, and there were few teacher explanations. The students were engaged in personal constructive mathematical activity and there were no management or organisational programs created by the program. Observation of individual students and interviews confirmed these impressions and indicated that teaching based on open-ended questions is suitable for both students who are confident at mathematics and for those who lack confidence. It was noted that even though the teaching and the classroom were structured to encourage co-operative work the student seemed to prefer to work by themselves most of the time.

Students showed significant improvement on a test of closed items based on the content of the program and the improvement was maintained after the program. Students also showed an overall improvement on the open-ended test questions although the tendency to give multiple correct responses was not maintained after the program.

Introduction

It is now taken as conventional wisdom among educators that learning opportunities will be maximised when students:

- are actively engaged in mathematical explorations;
- build on their existing knowledge and experience;
- are encouraged to develop and use their own strategies;
- feel confident to take risks and make errors; and
- are able to communicate with, and explain their ideas to, others.

The key elements seem to be that both teachers and students recognise the contribution of learners to the learning process, and that learners can have input into the solutions and solution paths. The imperative is to identify strategies for creating such learning opportunities within the constraints normally applying in classrooms.

One thing seems clear. It will not be possible to change the quality of learning by merely using conventional text exercises combined with new organisational strategies such as co-operative groups. It is necessary to identify tasks which can both satisfy our curricula goals and facilitate active learning. There is growing acceptance of the use of open-ended tasks as one way to engage students in the learning process.

Conventional teaching, particularly at upper primary and secondary levels, often consists of teacher demonstration of one or more exercises, with explanations and demonstrations linked to examples. Students predominantly work on drill and practice exercises. If the feedback is negative, students do more practice; if the feedback is positive then the class moves on to new exercises. Students are likely to see mathematics as a collection of rules and exercises.

An alternate perspective was explained by Christiansen and Walther (1986) who recommended that students work on tasks of the type:

> *Design a new playground (carpark) for your school.*
> *Compare the bounce of these four different balls.*

Lovett and Clarke (1989) compiled an outstanding collections of investigations of such realistic situations. Examples of their tasks include:

> *How many different houses made from four cubes can you design?*
> *How many people can stand on your classroom?* and
> *How long would it take for $1 000 000 worth of cars to drive past a point on your road?*

Krainer (1992) referred to the poles of the dilemma where on one hand we have mathematics as a complex and developed science, and on the other hand the need to acknowledge the spontaneity and creativity which students bring to their classes. Krainer (1992) described "powerful tasks" which are more than problems, but may deal with describing or discussing a situation. Powerful tasks are open-ended allowing the pupils to pose and discuss new questions. Among other features, the tasks stimulate a high level of acting and a high level of reflecting. One example of such tasks, taken from the Shell Centre (1985) is where students are asked to describe which of a variety of line graphs represent the way a student might hoist a flag on the school's flagpole.

Nohda (1986) recognised the connection between the task set and the type of mathematical thinking in which the learners engage. He described an "open-approach" method for teaching which combines both open-ended problems and problems for which there are multiple solution paths. He suggested that the open-approach teaching method fosters not only creative thinking in students but also mathematical activity at the same time. He presented an example where a diagram (the O and X represent different coloured marbles) is presented as follows:

O	O X	O X O	O X O X
	X X	X X O	X X O X
		O O O	O O O X
			X X X X

The students were asked to construct their own problems. Nohda reported that some of the problems the students constructed were:

> *How many marbles are there when each side of the square has 10 marbles? In such a square how many X's will there be?*

Nohda (1986) explained that the task is open in three ways. First, there is the openness in the students' activity. The main point here is that the questions are created by the students. This greatly contributes to the motivation to solve the problem. Second, there is openness in the mathematical content. Not only is the same mathematical potential present here as in text book tasks, there is also the possibility of generalisation and diversification of the problem. Third, there is the openness of interaction between the students and the mathematical content. In this Nohda contrasted conventional teaching where the teacher plans the lesson and approach beforehand with this mode where the students' problems and solutions are considered by the teacher and then used by the teacher as the basis of further tasks. He also noted that this approach caters for a range of abilities within the class.

Nohda (1986) listed the features of students' activity on such tasks as preliminary skirmishing, gestating, exploring, conjecturing, testing, explaining, reorganising, elaborating, and summarising. He noted that while conventional teaching does play an important part in mathematical development of students, it gives them a limited outlook.

Other examples of problems which he cited as suitable for this open approach are:

> *1) One of my classmates is sick now and has been in hospital. In the class meeting it has been decided that everyone will be give a card to the friend. Now we have to make some small rectangular cards (5 cm x 10 cm) from a larger card (45 cm x 35 cm). How many small cards can you make from the bigger one?*

> *2) We connect a house to another house by a direct telephone line. We put just one line between each house. Now make up your own problems and write them down (Nohda, 1991)*

In Nohda's tasks the openness arises primarily because of the variety of solution paths to a task and in the creativity necessary to invent their own questions. A similar approach was taken by Pehkonen (1992) who created "problem fields" out of which mathematically rich open-ended explorations are generated. A problem field consists of clear but rich problems from which many different explorations can arise. For example, the task:

> *With 12 matchsticks one can make a square with an area of 9 area units*

can be extended to include other investigations such as:

> *How many different polygons of 5 area units can you make with 12 matchsticks? Can there be more than ten different solutions?*

Such problem fields can stimulate the conditions for learning listed above mainly because after the initial tasks, the students can influence the direction and goals of the investigations.

Similar mathematically rich open-ended tasks were suggested by McGinty and Mutch (1982) They suggested open investigations involving patterns in decimals. For example,

> *Generate tables for \F(1,p) on a microcomputer and explore \F(1,p_{ten}) in other bases.*

Of course such an approach is not restricted to the teaching of mathematics. For example, Cliatt and Shaw (1985) used open-ended questions in science classes. An example was:

> *How do animals protect themselves?*

They argued that working on such questions goes beyond accumulating information and teaches students to think.

A dilemma with each of the approaches described is that teachers can be tempted to use such tasks as additions to the program, rather than as core activities. One approach which can help introduce open-ended tasks to mainstream teaching was described by Sullivan and Clarke (1988). They used the term "good questions" to describe a style of open-ended tasks which are also content specific. One example of a such an open-ended task is:

> *My vegetable garden is shaped like a rectangle. The perimeter of the garden is 30 metres. What might be the area of my garden?*

This question is different from conventional perimeter and area questions in two major ways. First, it requires a higher level of thinking and engagement than do conventional questions. Traditionally mathematical questions have required students to repeat a procedure or recall an algorithm. The sample question engages students in constructive thinking by requiring them to contrast the related concepts of perimeter and area and to think about relationships for themselves. Another advantage of the question over conventional items is that the need for thinking by individual students is made clear to the student. The students cannot rely on remembering a rule or simply manipulating formulas, they must think about the concepts, their meaning and the links between them. Further, Cobb (1986), Doyle (1986) and Desforges and Cockburn (1987) all noted the tendency for students to respond adversely to higher order tasks by seeking to have the demand of tasks made more explicit. These open-ended tasks have the potential to overcome this adverse reaction by stimulating higher level thinking within a specific framework.

Second, the question has more than one possible appropriate answer. Some students might give just one response, others might produce many appropriate answers, and there may be some who will make general statements. The openness of such questions offers significant benefits to classroom teachers because of their potential for students at different stages of development to respond at their own level (Sullivan, Clarke & Wallbridge, 1990).

Other examples of similar open-ended tasks are:

1) A number has been rounded off to 5.8. What might the number be?
2) Draw some triangles with an area of 6 sq. cm.
3) Find two objects with the same mass but different volume.
4) Describe a box with a surface area of 94 sq. cm.

In summary, there is increasing use of open-ended tasks as a way of encouraging students to become learners of mathematics by doing and even creating mathematics for themselves. Open-ended tasks have the potential to allow students to respond to questions in their own way, they offer teachers of heterogeneous groups a method for catering for the diverse ranges of interests and experience in the class, and they also allow the focus of the explorations to be mathematics. The next step is to examine ways in which such questions can be used in mathematics classes and to consider the responses of students to working on such tasks. This is one such investigation.

A classroom investigation of classroom activity based in the use of open-ended tasks

This is the report of a teaching program which was based as far as possible on the use of open-ended questions. The goal was to describe what happened when the primary activity in a mathematics class was the students' activity on open-ended tasks rather than the teacher's transmission of information. The teacher was a graduate student with 10 years teaching experience who had expressed an interest in the use of open-ended tasks. The class was at grade 6 (age 11, 12) level in an outer suburban primary school. While the community was predominantly working class, most families would have at least one employed adult and nearly all would speak English at home.

Since changes in learning, attitudes, or mathematical understanding generally develop over long periods examination of brief programs is difficult. In this case the data collection period was brief because of the breadth of data collected, but also because the teaching style was sufficiently different from common practice at the school that to extend the investigation may have been intrusive. It was hoped to learn about the effect on the teacher, the pupils, and their learning from a program based solely on open-ended questions.

One major concern was that the students would be unfamiliar with the style of the questions. To overcome this, the teacher was asked to use a range of open-ended tasks in each of the preceding topics. The observation period was 10 weeks into the school year. Prior to the program there was also a session during which the purpose of the program and the style of teaching were discussed. Examples of open-ended tasks from mathematics and other disciplines were given and there was discussion of the types of responses possible. The way these question differ from conventional exercises was also discussed with the students.

The goal of the investigation was to describe the outcomes of the class during such a program as fully as possible. Many reports of similar investigations emphasise the reproduction of transcripts of teacher/student and student/student interactions, and the presentation of the variety of solution paths used by students. as the primary data. It was assumed in the study that a rich interaction between teacher and students, and between students was a pre-requisite to this teaching approach, and that the tasks used in the program are sufficiently rich that they will by necessity stimulate different solutions and solution paths. Rather, this report of the investigation attempts to provide a broader picture of the class than that available through selected transcripts, and it attempts to represent the class setting using a combination of qualitative and quantitative reports.

The outcomes of the investigation are presented below The sections include a summary of the teaching program, report of the observations of the teacher, summary of the responses of students to closed and open-ended questions, some data on attitudes, and information from the observations of the students.

The teaching program

The focus of the program, length perimeter and area, was selected by the class teacher.

The program was arranged into 12 sections. These were called lessons even though each could last more than one mathematics period. The lessons were as follows:

Lesson 1	The concept of length and ways of measuring.
Lesson 2	Estimating length and choosing appropriate units.
Lesson 3	Using metres and centimetres, and introducing perimeter.
Lesson 4	Estimating and measuring perimeter.
Lesson 5	Area as covering.
Lesson 6	Area as covering.
Lesson 7	Linking area and perimeter.
Lesson 8	Linking area and perimeter.
Lesson 9	Area and perimeter of other shapes.
Lesson 10	Area and perimeter of other shapes.

A detailed program was given to the teacher. The information included a focus for each lesson, a selection of open-ended tasks, and some suggested conventional exercises.

Through this report just the details of events in lesson 3 are presented to illustrate the program and the outcomes. It would require too much space to report each lesson fully. Lesson 3 is representative of the other lessons and the study of one lesson illustrates how the data collected are inter-connected.

The program for the teacher contained the following information for this lesson:

Purpose	Key activities	Other activities
Lesson 3.	Draw a rectangle which is 12 cm around (*on squared paper*).	Measure width and height of specific objects using rulers in m and cm
Using metres, cm and mm., and introducing perimeter	Find something which is 10 cm long.	
	Find some things which are twice as long as they are high.	Estimate, then measure, a collection of lines, some straight, some curved
	Give the students a piece of string. Ask The string is the distance around some objects. What might some of the objects be?	Arrange a collection of containers (eg. bottles, jugs) by perimeter

Individual worksheets with space for written responses for each of the open-ended tasks were prepared prior to the program and presented in the format of a workbook. Advice to the teacher to facilitate student work on the questions, with a review or summary at the end of work on each task was prepared and discussed with the teacher.

Teacher Observation

The program was based on the premise that students learn from their own exploration and thinking about open-ended mathematical tasks. It was intended that the teacher have a somewhat less central role than in a conventional classroom. One aspect of the data collection was to examine the role of the teacher. The teacher was observed over nine lessons using a combination of naturalistic and structured recording. It was intended that there be two observers in each lesson; one to record teacher actions, and the other to focus on students, as is reported below. The observer had a proforma which facilitated the recording of the information.

The structured observations and recording were based on the principles of the SCAN notation (Beeby, Burkhardt & Fraser, 1980), although the focus was more specific. In particular, only the teacher's actions related to the categories of questions asked, tasks set, and explanations given were recorded. Teacher and students' questions, and specific instructions to perform a particular action (eg. draw a triangle) were coded as questions. The activities, problems and exercises which were intended for the students to work on either individually or in groups were coded as tasks. The explanations arose when the teacher sought to explain the mathematical concepts involved in the tasks or questions.

The questions and tasks were coded on a two dimensional grid, with open/closed on one axis and thinking/remembering on the other. While this is somewhat blunt, the more specific coding in the SCAN system is difficult to use and requires many observations for meaningful trends to appear. A question was considered open if there was more than one appropriate response possible. A question was coded as remembering if it could be answered by merely recalling previous information. The following are examples of coding of four questions from the program:

	Closed	Open
Remembering	What is 5 times 5?	Draw a rectangle.
Thinking	What is the area of this shape?	What can go wrong when you are measuring area?

The explanations were coded as either relational, instrumental, or other. Relational explanations were those which seemed designed to stimulate the quality of understanding which Skemp (1976) termed relational. An example of an explanation which would be coded as relational is "To find the area of a rectangle, work out how many square units would be needed to cover the rectangle". The explanation "... multiply length times width" would be coded as instrumental. If the explanation fitted into neither category it was coded as "other".

The observer was trained over two sessions. The coding system was explained, then lessons from a video of a series of primary school mathematics lessons were viewed, one by one, using three observers, with a comparison and discussion of coding after each lesson. The tape was reviewed where there was any disagreement. There was a second session with the observer and the researcher reviewing video lessons. There was a high degree of consistency of the coding of the aspects of the lesson events which are of interest here.

The actions of the teacher were recorded in nine lessons. Table 1 presents the overall total of the codes for the questions and tasks in the various categories in these observed lessons:

Table 1: No. of questions and tasks coded over nine observed lessons.

	Questions		Tasks	
	closed	open	closed	open
remembering	79	23	5	5
thinking	39	115	0	17

Given that it is common for closed remembering questions to represent about 60% of all classroom events (Sullivan & Leder, 1991), this breakdown represents a major departure from conventional lesson structure. Nearly half of all question were coded as open-ended thinking questions and two thirds of all tasks were coded in that way.

There were 45 explanation events overall; 33 were coded a relational, 10 as instrumental, and 2 as other. Given that an explanation event may be a brief as a single sentence, this represents an average of only five explanations per observed lesson.

Overall it seems that the implementation of the program was compatible with the intention which was that the questions and tasks would be open-ended and require creative input from the students, and that there would be few teacher directed explanations.

There were a range of additional data collected to provide an impression of the content and style of the lessons. These included:

- an unstructured summary of the teacher's actions by the observer for each lesson;
- a diary completed by the teacher daily to record her reflections; and
- recording of the lesson review.

To allow consideration of the style of the lessons and the data collected, the following is the report on lesson 3.

The unstructured recording of the development of the lesson was as follows:

9:32 *Teacher hands out worksheet 3.1.* (This consisted of a sheet with the questions "Draw some rectangles which are 12 cm around" and the rest of the page was 1cm grid paper). *The children look at the sheet.*

9:35 *The pupils get to work on 3.1.*

9:41 *The teacher instructed them to finish work and watch the front. There was a review of 3.1. Teacher gets different children to give answers. Puts two correct answers on the board.*

9:49 *Asks them to name their work.*

9:50 Asks them to look at sheet 3.2 (This had two questions; What are some things which are 10 cm long? and What are some things which are twice as long as they are high?) They start working. The kids worked fairly productively.
10:08 Review of sheet 2. One or two examples.
10:10 The class moves on to sheet 3.3 (The children have a length of string. "Your string is the distance around some objects. What might some of the objects be?"
10:17 They go back to seats.
10:20 Looking at the last page (Estimating and measuring lines)
10:28 The class stops for review.

In this lesson 35 questions asked; 21 were coded as open thinking, and 11 as closed recall. Some examples of the open thinking questions asked were:

Describe one of your rectangles.
How do you know that gives you 12?
Why is this not right?
What makes up a rectangle?
Have you got another one which goes to 12?
What could you do to prove that it did work?
Who found something different?
Who was able to find two objects which look quite different but which are the same distance around?
How will we know where you measured it?

In the lesson there were 7 explanations, 6 of which were recorded as relational.

After each lesson the teacher recorded reflections in a diary. The following is the extract for this lesson. This lesson was held after problems; one because some students from the class had started a fire in the school playground before school, and another because parents had complained about the content of a recent sex education lesson.

> *After the hectic morning had subsided, chn didn't seem to be adversely effected and jumped into their maths mode.*
> *In the initial review, actually just clarifying the point of contention in which some children were counting 12 squares, other drawing 12 X 12 cm square shapes and the most basic questions: What's the difference between a square and a rectangle?*
> *The class found 2 rectangles with 12 cm perimeter. I should have been more organised and used a proper unit in my modelled b/board work as I'm sure the question about square/rectangle arose as a query of one of my diagrams.*
> *Chn did not seem to notice any pattern so I was not going to force my point. I'll wait until lesson 4.*
> *Again I felt that had I given more opportunities for chn to practise/explore other perimeters - the chances of recognising patterns would have become evident*

> *I'm probably still trying to do too much in this session and enhancing some levels of frustration in some chn.*
> *Probably not giving them enough time to come up with multiple responses is not helping either.*
> *I could be challenging chn's responses more thoroughly and encouraging other chn to commit themselves to a strategy being discussed and then encourage them to find flaws or refine the methods.*
> *Had I used correct units when reviewing strategies with chn - Fiona & Vince's strategy of checking "twice as long as high" would have definitely come clearer to the other chn*
> *The string we're using isn't the best - some chn are finding it stretches significantly.*

It is clear that the teacher is attempting to rely on the students' responses and thinking, but she is also feeling a real tension that there are children who were experiencing difficulty. She was concerned that the pupils did not identify the patterns, and was tempted to use more examples to draw this out.

There were similar data collected for each of the lessons. These data illustrate that the program as implemented was directly focused on the open-ended tasks. The teacher made no attempt to transmit information or rules, and relied on the pupils' report of their activity in the review.

Written test responses

One of the measures of the outcomes of the program was from the students responses to two written tests which they completed before the program (pre-test), after the program (post-test), and again three months after the teaching program (delayed post-test). The results of this testing are not presented here due to space limitations, but overall the suggest that the students may have learnt the basic concepts as an outcome of their activity on the open-ended tasks, and that this was retained and even extended over time. On the other hand, the tendency to give multiple responses diminished on the delayed post-test suggesting that it was not the normal mode for their responses.

Observation of students

Within this teaching study, to investigate the effectiveness of open-ended tasks for stimulating learning, there was a focused study of the learning of selected students. This was basically a case study of these students' behaviours within the controlled learning environment.

Six students were chosen by the class teacher; two rated as high achievers, two as "average" and two who experience difficulty with mathematics. The researchers were not told which student belonged in which category.

The features of the study of the six students were:
- a record of the type and level of engagement during the lessons;
- a study of their contribution to class reviews;
- review of their written work on the worksheets and on each of the six tests;
- examination of diaries or journals which the students completed after each lesson;
- structured interviews with the students after the post-test.

The details of these data are reported in Bourke (1993) and Scott (1993). This report presents only a summary of the data on the engagement of the target students and also brief summaries of the impact of the program on two of the six students.

The level of engagement.

One of the key claims about the use of open-ended tasks in classrooms is that children become more involved in their own mathematics learning. This perhaps suggests that students would be more obviously engaged on class tasks. It has also been suggested that the time spent in learning is an important determinant of what is learnt (Berliner, 1978). Both to assist in the evaluation of the overall program, and to provide a focus for the study of the impact on learning, a record was made of the type of activity of the selected students.

An observer recorded the type of activity in which each of the six target students was engaged using one of seven pre-determined categories. Observations were made in 10 of the lessons, with a record made every two minutes for each student. Table 7 presents the breakdown of the number of each type of behaviour for each of the six target students.

There are a number of important observations. It was anticipated that the category, "working with a partner/group" would be the main activity in which the students would be engaged, yet it comprised only 5% of all recorded observations. Nearly 40% of observed events were of the students working by themselves. This was a most surprising result. The teacher had arranged the class to facilitate group work. The seating in the room was in groups, not oriented toward the front of the classroom. The impression of the observers was that most of the work was being done in groups. Certainly the students could have worked with a partner or a group as often and as much as they chose. It raises an issue of what constitutes group work. Two students sitting together, doing the same task and talking about the task on occasion may achieve the advantages of co-operative work, yet the main activity may well be individual.

Table 7. Breakdown of type of classroom observation for each of the six target students

Type of activity	Target Students A	B	C	D	E	F	Total	Percent of total events
Daydreaming	8	15	34	11	5	9	82	6%
Off-task chatter	9	0	10	20	10	15	64	4%
Other off-task behaviour	17	22	28	37	5	70	179	12%
Listening/talking to the teacher	99	54	78	74	51	69	425	29%
Working by self	114	106	106	107	53	86	572	39%
Working with partner/group	14	8	7	13	10	16	68	5%
Organising for the task	9	12	8	11	1	7	48	3%
Waiting	4	2	3	1	4	2	16	1%

Not all students were present for each of the ten lessons.

It was also surprising that even though the goal of the program, the teacher's intentions, and even the researchers' impressions suggested that the time listening to the teacher was minimal yet it represented in nearly 30% of observations. As was noted earlier, the number of teacher explanations was low, and so the figure may well refer to the time in which students were explaining and describing their own solutions to the tasks to the whole class with the discussion led and facilitated by the teacher.

If we combine the three categories "listening to the teacher", "working by self" and "working with partner ..." to signify on-task behaviour, this is 80% of all observations. There are no control data available of how the code would apply to conventional classes, but the impression is clearly of students who are engaged in learning activity predominantly by themselves with significant opportunities for participation in class reviews.

Note also that the distribution of the type of activity was similar for each of the target students, even though they represent a cross-section of abilities at mathematics.

Additional information about the target students was determined in two associated studies. Scott (1993) examined the learning of target students by integrating the data described here with a study of learning environment preference of the students based on interviews, selection of preferred learning situations for pre-set list, and from ranking of the perceived effectiveness of different learning situations presented via photographs. Bourke (1993) studied the key stages in learning by examining and integrating the data available on the program and the class with the classwork and test responses of the target students, and post-test interviews.

As with many case studies, the reports on these target students may help to clarify some of the surprise results mentioned above. One report of a student, called Abagail here, gives some insight into a possible explanation of the apparent ease of the open-ended tasks, and also into the apparent low frequency of the use of group work.

Abagail was popular and competent at most aspects of the curriculum, but lacked confidence in, and was confused by, mathematics. She reported that she expected to find mathematics difficult and irrelevant. Abagail showed a marked improvement as a result of the teaching program. On test 1, her scores on the pre-test, post-test and delayed post-test were 4, 2 and 5 respectively. The corresponding scores for test 2 were 1, 5 and 5. Two aspects of her responses to interviews and other instruments help to explain these trends and give some insight into the results presented above.

First, Abagail made it clear in interviews that she much preferred items placed in some real world context. She was asked to compare two given questions and to state which one she preferred and to give reasons. The following is an extract of an interview:

Interviewer: Why did you choose "On TV the reporter said that there were 32000 people starving in Somalia. What might the exact number be?"

Abagail: 'Cos at least you know what they are talking about... They give you some info.

I: So what info would you like to know in this other one? ("Round there numbers to the nearest 10 000")

A: I don't know. Maybe what the numbers are about.

Abagail indicated a similar preference on 4 out of 5 such choices. Abagail's positive journal entries dealt with the times in which she had a knowledge of the context in which the tasks were set.

As it happens, most of the closed items on test 1 were context free and were of the style "Draw a rectangle ..." The open-ended questions were generally set in some context. It may be that the context was more helpful than the openness.

While it is easier to present open-ended questions in some context, and easier to make context questions open, the real cause of the explanation of the better result on the post test may still require further elaboration. Quite possibly the combined effect of the open-endedness and the real world context assisted the students.

Second, Abagail, in interviews, and in the choice of tasks and ranking of photos, indicated a preference for learning in small group situations under the guidance of the teacher, and for opportunities for practice and review. This was indicated as potentially beneficial by Nohda (1991) and Christiansen and Walther (1986). The improved score on test 1 of the delayed post-test may be an outcome of the consolidation of the information in her mind.

Another of the target students is called Bernard here. His case is also illuminating. Bernard was talented in many areas of the curriculum, he was artistic and a lateral thinker. He was independent and did not enjoy simple tasks. His scores of the test were well above the class average, and both post-test and delayed post-test scores on both tests were nearly perfect. Bernard gave multiple responses on the post-test to four out of the six items indicating that he could interpret the full implications for the questions. It is clear that he had a high rating of off-task behaviour during the early lessons, particularly during the reviews. However during the later lessons with more difficult tasks, Bernard was much more engaged.

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References

Australian Council for Educational Research (1984) *PAT Test Series.* Hawthorn, author.

Beeby, T., Burkhardt, H., & Fraser, R. (1980). *Systematic Classroom Analysis Notation. Nottingham,* Shell Centre, University of Nottingham.

Berliner,D.C. (1978). *Allocated time, engaged time, and academic learning time in elementary school mathematics instruction.* Paper presented at the 56th Annual Meeting of the National Council of Teachers of Mathematics, San Diego, April 1978.

Bourke,D. (1993).Exploring content-specific open-ended tasks: Three case studies. Unpublished M.Ed. Thesis, Australian Catholic University.

Christiansen, B. & Walther, G. (1986) Task and activity. In B. Christiansen, A.C. Howson, & M. Otte (Eds.) pp.243-307. *Perspectives on Mathematics Education.* Holland: Reidel.

Cliatt, M. J. & Shaw, J. M. (1985). Open questions, open answers. *Science and Children,* 23(3), Nov-Dec., 14-16.

Cobb, P. (1986). Contexts, goals, beliefs and learning mathematics. *For the Learning of Mathematics,* 6(2) June, 2-9.

Desforges, C. & Cockburn, A. (1987). *Understanding the mathematics teacher: A study of practice in first schools.* London: The Palmer Press.

Doyle, W. (1986). Classroom organisation and management. In M.C. Wittrock (ed), *Handbook of research on teaching.* 392-431. New York: MacMillan.

Feldman, D. S. & Gagnon, J . (1986). *Statview 512+.* Calabasas CA: Brainpower.

Krainer, K. (1992). *Powerful tasks: Constructive handling of a didactical dilemma.* Proceedings of the sixteenth conference of the International Group for the Psychology of Mathematics Education (Volume 2, 2-25), New Hampshire.

McGinty, R. L. & Mutch, W. (1982) Repeating decimals, geometric patterns and open-ended questions. *Mathematics Teacher,* Oct, 600-602.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA.: The author.

Nohda, N.(1986). A study of open-approach method in school mathematics. Tsukuba Journal of Educational Study in Mathematics, 5, 119-132

Nohda, N. (1991). Paradigm of the "open-approach" method in mathematics teaching:Focus on mathematical problem solving, Zentralblatt fur Didaktik der Mathematik, 2,pp.32-37

Owen, E. & Sweller, J. (1985). What do students learn while solving mathematical problems? *Journal of Educational Psychology,* 77(3), June, 272-284.

Pehkonen, E. (1992). Using problem fields as a method of change. The Mathematics Educator, 3(1), 3-6

Putnam, R., Lampert, M. & Peterson, P. (1989). Alternative perspectives on knowing mathematics in elementary schools. In *Review of Research in Education,* 16, (pp. 57-149). American Association for Research in Education.

Scott, A. (1993). A study of learning environment preferences of two year six students. Unpublished M.Ed. Thesis, Australian Catholic University.

Shell Centre (1985). *The language of functions and graphs.* Shell Centre for Mathematics Education. Manchester: Joint Matriculation Board.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77,* Dec., 20-26.

Sullivan, P. & Clarke, D. (1988). Asking better questions. *South East Asian Journal of Science and Mathematics Education,* XI (I), June, pp.14-18.

Sullivan, P. Clarke, D.J.& Wallbridge,M. (1991) Problem solving with conventional mathematics content: Responses of pupils to open mathematical tasks. Mathematics Teaching and Learning Centre Report No. 1.

Sullivan, P. & Leder. G. (1991). Questions and explanations in primary mathematics teaching: A classroom investigation. In P. Sullivan and D. Clarke, *Communication in the Classroom* (pp. 51-60). Geelong: Deakin University Press.

Teaching children to think mathematically

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Background

> Thinking is the means used by humans to improve their understanding of, and exert some control over, their environment (Burton, 1984, p36).

Thinking mathematically is not, therefore, an end in itself. Rather it is a process through which we make sense of our world (Mason et al, 1982, p178). Thinking as sense making is deeply embedded in the constructivist viewpoint in which the learner is considered as an active purveyor of meaning (McGuinness, 1993). Within this tradition, a clear distinction may be drawn between mathematical thinking and the knowledge base, strategies and techniques described as mathematics. This distinction has often been expressed in dichotomies such as process versus product (eg: Burton, 1984).

A growing emphasis on the processes of mathematics during the 1980s combined with a prevailing trend towards constructivist views of learning led to a perception that mathematical thinking skills should be learned in the context of open ended problems.

But what does mathematical thinking actually consist of? Axioms, theorems, proofs, definitions, hypotheses, formulae, algorithms etc are all essential elements of mathematics, but none of them is at the heart of the subject. At its heart, mathematics is about solving problems and the function of mathematical thinking is to make sense of problem situations from a mathematical perspective using mathematical tools; eg: Burton's (1984, p38) processes of mathematical thinking: specialising, conjecturing, generalising and convincing.

Learning to think mathematically, however, is more than just learning to use the tools of mathematics, although developing a facility with the tools of the trade is clearly an element. Mathematical thinkers have a way of seeing, representing and analysing their world, and a tendency to engage in the practices of mathematical communities.

> Learning to think mathematically means (a) developing a mathematical point of view - valuing the processes of mathematisation and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade and using those tools in the service of the goal of understanding structure - mathematical sense making (Schoenfeld, 1994, p60).

Whilst mathematical sense making may have its roots in constructivism, developing a mathematical point of view may be more akin to enculteration into a community. But which community? Historically, pure ideas of the mind have been held in higher esteem in education than those which might be seen as of use in commerce or engineering . The debate continues to this day between those who see mathematics as a form of high culture, emphasising abstract algebra and formal proof, focused inwards on itself (eg Gardiner in Neumark, 1995) and those who see it as a practical and useful human construction gaining its status from its power to explain, organise and change our world (eg: Burton, 1994).

The privileged position occupied by mathematics in the national curricula of most countries, however, is due in no small part to its perceived usefulness. Unfortunately, since the 1970s, it has become increasingly clear that employers, school inspectors and others are dissatisfied with the inability of school leavers to transfer the knowledge and skills taught in mathematics lessons to other contexts in the workplace and in everyday life (eg DES, 1979). Research has shown (Lave, 1988; Nunes, Shliemann & Carraher, 1993) that in "real life" situations school learned procedures are unlikely to be used. In fact when Lave (1988) compared adults' use of mathematics in test situations with the techniques employed in a shopping context making similar demands she found that, not only did the technique chosen depend on the context, but the similarities which existed between the situations were not perceived.

Transfer of learning

Lave (1988) suggests that knowledge is "situated", implying that all knowledge is linked to the context or situation in which the learning occurred and that the notion of transfer is impoverished. If cognition is socially situated and mathematical knowledge learned through apprenticeship rather than individual construction, how could transfer occur? Knowledge might not be transferable, but limited to particular social contexts. If this is the case, then hard questions must be asked about the utilitarian nature of school mathematics. To what extent can mathematical thinking skills be considered transferable?

Research has often shown that even when students have appropriate knowledge and techniques, it is not automatic that they will apply them. It is usually stated that the student must choose to use that knowledge (Silver, 1987). Resnick (1987) claims that learning to recognise, or even learning to search for, opportunities to apply one's knowledge is an essential skill. Prawat (1989) suggests that applying knowledge requires the student to recognise that previous learning might apply, retrieve the knowledge and transform it to match the new situation. These metaphors would be rejected by Lave (1993) as artificially separating knowledge from its context, the learning mind from the world, and thoughts, feelings and values from their cultural-historical source.

Many writers suggest some form of conscious control of the problem solving process eg: executive (Schoenfeld, 1985) or inner monitor (Mason & Davis, 1991). These are aspects of metacognitive processes (Flavell, 1979; Palincsar & Brown, 1984; Gray, 1991) which are used to plan, monitor and control the problem solving process, and attempts have been made to integrate metacognitive processes into theories of modelling and applied problem solving (Lester, 1985; Schoenfeld, 1987; Tanner & Jones, 1994).

The social world of the classroom may restrict or encourage strategy selection, however:

> The social... invisibly pervades even situations that appear to consist of individuals engaged in private cognitive activity... mental work is rarely done without the assistance of tools... Cognitive tools embody a culture's intellectual history; they have theories built into them, and users accept these theories - albeit unknowingly - when they use these tools (Resnick, 1991, p7).

Not all strategy choice is made through conscious rational control (Siegler, 1991, p241). Research suggests that children associate particular strategies with particular problem contexts and that the use of such strategies is resistant to change (Siegler, 1991, p242). Automatic processing dependent on context seems to be the preferred option, with conscious control used only when all else fails.

> People interpret the world rapidly, effortlessly. But the development of new ideas, or evaluation of current thoughts proceeds slowly, serially, deliberately. People do seem to have two modes of operation, one rapid, efficient, subconscious, the other slow, serial and conscious
> (Norman, 1986, p542).

The significance for learning and problem solving is that there seems to be an implicit mode of thought which is swift and effortless but inaccessible and another more explicit mode in which there is greater access to the rules and knowledge used but processing is more effortful and less efficient (Halford, 1993, p47).

The greater efficiency of automated processes suggests that learning ought to aim at their development, but the dual nature of processing suggests that more than implicit knowledge is required for applied problem solving. Understanding and therefore explicit knowledge is required in order to transfer knowledge to new tasks, and in order to reorganise domains of knowledge and relate them to other domains. The transition from implicit to explicit knowledge is not inevitable, requiring considerably greater cognitive effort; and partly explains the tendency of many students to avoid the process (Halford, 1993, p47-50). Many teachers also avoid the process and emphasise low level learning and practice of standard cases rather than aiming for the higher prize of transferable mathematical thinking skills. Transfer is not a well defined construct however and we can distinguish between "low road" transfer resulting from relatively automatic generalisations based on much continuous practice and "high road" transfer attained through mindful abstraction (Salomon & Globerson, 1987, p624).

Low road learning underlies socialisation and learning based on experience, social reinforcement etc. High road learning is more typical of explicit instruction aimed at activating non-automatic strategies with materials that require mindful abstraction, consideration of alternatives etc. The low road only facilitates close transfer whereas high road facilitates far transfer.

Piagetian notions of reflected abstraction are enhanced by the concept of mindfulness, defined as:

> the volitional metacognitively guided employment of non-automatic, usually effortful processes. Mindfulness is a mid-level construct which reflects a voluntary state of mind, and connects among motivation,

> cognition and learning. It is both a general tendency and a response to situational demands. (Salomon and Globerson 1987, p623)

But how is such "mindfulness" learned? Lave et al (1988) distinguish between apprenticeship models of learning and school learning, claiming that often school based learning has only the "veneer of learning" and suggests the introduction of an apprenticeship model in which students "come to think like good mathematical thinkers". This might involve: finding and comparing patterns, varying problems, inventing problems and long investigations which might sometimes have more than one solution, considering their understanding and how they came to it, using the world to provoke mathematics and teaching other children (Lave et al, 1988, p74-79). This sounds very much like high road learning and the development of metacognitive awareness.

She takes a Vygotskian position on the nature of learning in that:

> Learning is a process that takes place in a participation framework, not in an individual mind. This means, among other things, that it is mediated by the differences of perspective among the co-participants. It is the community, or at least those participating in the learning context, who learn under this definition (Lave & Wenger, 1991, p15).

This may be an adequate metaphor for the social processes at work during low road learning or the development of automatic processes, but is inadequate to describe the complex relationship between the learner and his/her social situation during high road learning and mindful abstraction.

Apprenticeship models of learning do not tell the whole story. Learning is a highly complex activity which, although socially situated and involving induction into communities of practice, also involves an individual search for meaning. Individual construction also has a part to play.

Bristow and Desforges (1995) offer case studies of two seven year old girls engaging in a negative numbers activity using the metaphor or context of a lift in a block of flats. One girl, Zoe, focused on performance and the social interaction with the interviewer throughout. The other girl, Sarah, finding that the task challenged her existing understanding, regarded the task as an opportunity to learn and actively sought to seek connections with existing knowledge to develop personal meaning. It is suggested that Zoe's learning is likely to be situated whereas Sarah's metacognitive awareness and mindful abstraction is likely to lead to transferable or applicable knowledge.

Our social constructivist viewpoint leads us to suggest that knowledge might be learned in transferable form if classroom communities of practice can be structured so as to encourage metacognitive awareness and mindful reflective abstraction.

The Mathematical Thinking Skills Project

The Mathematical Thinking Skills Project was funded by the Welsh Office and the University of Wales 1993/4 and aimed to develop and evaluate a mathematical thinking skills course. The course was based on activities developed in the Practical Applications of Mathematics Project (Tanner & Jones, 1993a, 1994). The National Mathematics Curriculum in England and Wales requires pupils to hypothesise and test, to generalise, and to prove their conclusions. The project aimed to accelerate the development of such formal modes of thought by enhancing key metacognitive skills such as planning, monitoring and evaluating in the context of open problems.

The course targeted metacognitive rather than cognitive skills. It was expected therefore that "close transfer" would be achieved and that the metacognitive skills of students in intervention classes would be enhanced. It was assumed that students would apply their newly acquired thinking skills to any mathematics which they met subsequently thus learning it in a qualitatively different way. "Transfer at a distance" into the cognitive domain was not expected to be immediate, therefore, but as new topics were met. Thinking skills pay for themselves not so much during the week in which they are acquired but during the years that follow (Perkins 1987).

Methodology. An action research network of six secondary schools was established, drawing students from a variety of social and ethnic backgrounds. The schools developed and trialled teaching strategies and materials, supported by members of the project team. The sample was not random due to the degree of commitment demanded from the teachers involved and consequent difficulties of self selection.

Two teachers from each school, who were to be involved in teaching intervention lessons, attended an initial one day induction course to familiarise them with the theoretical underpinning to the project and the outcomes of previous work, in particular, effective teaching strategies. They then attempted to integrate these approaches into their own teaching styles.

Intervention lessons were led by normal class teachers rather than outside "experts". The advantages of this approach in terms of realism, pupil-teacher relationships and teacher development are clear. The approach carried the disadvantage, however, that the experiences of the intervention classes were not standardised. Regular participant observation by the university researchers was necessary to record the nature of the interventions made. These observations revealed that the extent to which teachers were able to adopt the approach was variable. In one case at least, the attempt to marry contrasting styles resulted in confusion. In another case a traditional outlook overcame the novelty of the materials and a completely didactic approach was employed. Purely quantitative approaches often fail to see the realities of classroom interaction. Qualitative data added some necessary illumination.

Two matched pairs of classes were identified in each school to act as control and intervention groups. One pair was in year seven (11-12 years old) and one pair in year eight (12-13 years old). Matched classes were either of mixed ability or parallel sets in every case.

Written test papers were designed to assess pupils' cognitive and metacognitive development. The sections of the test designed to assess cognitive ability were based on a neo-Piagetian structure and items were classified as identifying one of four stages of development, which were referred to as: early concrete, late concrete, early formal and late formal. Items were placed in the context of four content domains: Number, Algebra, Shape and Space, and Probability and Statistics. Items emphasised comprehension rather than recall. Classification took account of the anticipated memory requirements, National Curriculum assessment, and the results of large scale studies such as the Concepts in Secondary Mathematics and Science Project, (Hart, 1981).

The metacognitive skills of question posing, planning, evaluating and reflecting were assessed through a section in the written paper entitled "Planning and doing an experiment". Metacognitive skills of self knowledge were also assessed by asking students to predict the number of questions they would get correct before and after each section. In addition to the written papers, the metacognitive skills of a sample of 48 pupils were assessed through one-to-one structured interviews. These were conducted whilst the pupil planned and carried out an investigation into the mathematical relationships inherent in a practical task.

The pilot course and intervention teaching lasted for approximately five months. Regular network meetings were held at which experiences were exchanged, strategies discussed and new activities devised and refined. Post-testing occurred at the end of the course. Delayed testing occurred four months later.

The Thinking Skills Course. There were two strands to the course:

- the development of a structured series of cognitive challenges to stimulate the progressive evolution of key skills in the areas of strategy, logic and communication;
- the use and development of teaching techniques which would encourage the maturation of the metacognitive skills of planning, monitoring and evaluation.

Underpinning both strands was a continual emphasis on the need to explain rather than describe, to hypothesise and test, and to justify and prove. Metacognitive skills were not taught through the content of the materials but through the teaching approaches used (Tanner & Jones, 1995a; 1995b), which tried to develop skills of planning, monitoring and self evaluation and, by so doing, encourage students to construct and evaluate their own strategies through discussion and debate. Teachers encouraged students to think and plan for themselves and discuss their work, but they were not afraid to intervene to guide discovery.

The activities in the course did not address directly the questions used in the test of cognitive ability used to evaluate the success of the course. We were not "teaching to the test" but were hoping to establish "transfer". The results indicate that the course was successful in developing transferable knowledge (Tanner & Jones, 1995b).

The Results

Pre-tests. The assessment paper was trialled with 60 pupils from a school not involved in the project. Analysis indicated that the test was reliable (Cronbach's alpha = .86), and internally consistent for cognitive and metacognitive abilities. Correlations between assessments of cognitive and metacognitive ability made through interview and written paper confirmed that metacognitive and cognitive abilities were very closely linked ($p<.001$).

T-tests on the pre-test data showed no significant differences at the 5% level of significance between control and intervention groups for scores on the attitude questionnaire, the test, or its cognitive and metacognitive sections.

Post-tests. Covariate analysis of the overall test results using pre-test scores as covariates showed a significant difference in favour of the intervention groups at the 0.1% (0.001) level. Analysis of metacognitive skills showed improved performance by intervention classes and little change in control groups. These differences were significant at the 0.1% (0.001) level. Improved performance in the cognitive test for intervention groups was significant at the 5% (0.05) level.
Following the analysis of the post-tests, teachers were invited to comment on the results:

> *"Sue":* I definitely think it has helped their thinking skills. I said at the beginning that if you could convince me you could convince anybody because I was completely against it but now, I definitely can see the worth of it.

In the new classes formed for the new academic year some of the teachers now had students from both intervention and control groups. They were convinced that there was a marked difference between such students:

> *"Ann":* Well, the content that they were taught by us last term was exactly the same, both classes have done the exactly same work. But looking at the work this term, the intervention class metacognitively, planning and evaluating and that, the intervention class are, no doubt at all, far better. I have had much better work in from that half of the class - I've got the best of both classes now in the top set in year 9 from the intervention and control groups in year 8.

> *"Sue":* Test and homework results this year so far are better from the students from last term's intervention class. They seem to be able to think more clearly.

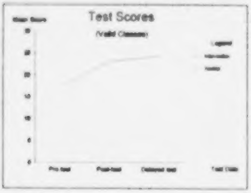
An improvement in algebraic skills was noted in both the ethnographic and statistical data. Teachers reported a greater willingness on the part of intervention pupils to generalise with letters:

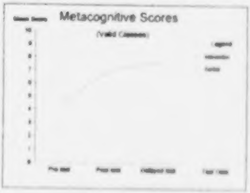
> *"Ann":* In investigations they have been far more adventurous in trying to use algebra but they were taught formulas in exactly the same way as the other class.

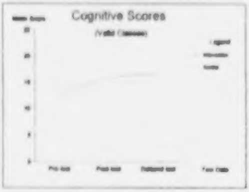
Such comments corroborate the statistical findings.

Delayed tests. The graphs of test scores for the valid schools show how the gap which opened up between intervention and control classes was sustained after the end of the course. Intervention students continued to progress in parallel with control students but at a higher level.

Covariate analysis of the delayed test results using the pre-test scores as covariates showed a significant difference between control and intervention classes at the 0.1% (0.001) level for the test overall and the metacognitive sections, and at the 5% (0.05) level for the cognitive sections. A sustained improvement in mathematical performance was observed. The improvement was sustained in both metacognitive and cognitive aspects.







Acculturation and the conditions for transfer

What do we learn when we learn mathematics? To an extent, the cultural theorists are right when they claim that we learn how to behave amongst school mathematicians. In a very real sense we are inducted into a community of practice in which certain behaviours are valued and rewarded whereas others are rejected.

The form of individual constructions is necessarily influenced by the tools which are brought to bear on it and these tools necessarily "embody a cultural history" (Resnick, 1991, p7), but this does not mean that the constructions are completely determined by the social context. Individuals bring a range of intellectual histories and social identities to school and in a real sense are preformed although not precast. Their identity in the new context may well be constructed from within the context, but on the basis of that preformation. Taking a middle position between individualistic and collectivist perspectives, one might claim

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that "the teacher and students interactively constitute the culture of the classroom" through negotiation and communication (Bauersfeld, 1994). But within the constraints of that negotiated culture, individuals construct.

During the project, different groups of students reacted very differently to each other when presented with open practical problem solving situations. Supposedly "open" situations are not as open and free as they seem. To learn mathematics is not simply to learn a list of skills and routines. To learn mathematics is to take part in a process of socialization, or acculturation (Schoenfeld, 1987). Acculturation is the process of generating a consensual mathematical reality, through social interaction, which constrains individual ways of thinking but does not determine them.

Our students need to negotiate ways of acting and thinking which empower them to explain, organise and change their worlds, not just operate within the society of the traditional mathematics classroom. Unfortunately many students associate acceptable ways of acting and thinking mathematically with dependence and repetition (Lester & Kroll, 1990, Gray, 1991). The consensual mathematical reality in many classrooms which defines the nature of objective, socially acceptable mathematics denies independence, initiative and creativity. The knowledge gained in such communities of practice is likely to be heavily situated (Tanner & Jones, 1994).

Fortunately, the converse also seems to be true - that communities of practice can be established which encourage the construction of meaningful knowledge and the use of non-automatic strategies and mindful abstraction. The ethnographic data collected during phase one of the project led us to the conclusion that some teachers were far more successful than others in creating the social conditions necessary for open ended problem solving skills to be learned in a transferable form (Tanner & Jones, 1993a, 1994). The results described here suggest that when teaching approaches encourage the development of metacognitive skills such as planning, monitoring, evaluating and reflecting, mathematical thinking skills can be learned in a transferable form, leading not only to an improvement in open problem solving but also to improved performance in learning the subject base or content of mathematics (Tanner & Jones, 1995b).

In order to transfer knowledge to tasks and to reorganise domains of knowledge to relate them to other domains an individual must engage in effortful mindful abstraction - a process of individual construction rather than socialisation. Students who have rarely been asked to engage in such effortful construction will probably be both unskilled in the process and unaware of its potential in the wider world. They may even consider that such processes are not valid within the mathematical domain, taking the position that mathematics is about rule following rather than a means of explaining, organising and potentially changing their worlds.

The metacognitive skills of planning, monitoring, evaluating and reflecting seem to form part of an infrastructure of mental habits which support both the transfer of knowledge to new situations through the techniques of mathematical modelling and the learning of new mathematics.

Literature

Bauersfeld, H. (1994). Theoretical perspectives on interaction in the mathematics classroom. In R. Biehler, R. W. Scholz, R. Straesser, & Winkelmann (Eds.), *The didactics of mathematics as a scientific discipline* (pp. 133-146). Dordrecht: Kluwer.

Bristow, S., & Desforges, C. (1995). Working practices and children's application of subject knowledge in the primary school. *Research papers in education, 10*(2), 273-293.

Burton, L. (1994). Clashing epistemologies of mathematics education: Can we see the "wood" for the "trees"? *Curriculum Studies, 2 No.2*, 203-219.

Burton, L. (1984). Mathematical thinking: the struggle for meaning. *Journal for research in mathematics education, 15*(1), 35-49.

DES (1982). *Mathematics Counts*. Cockcroft, W. H. (ed), London: HMSO.

Flavell, J. H. (1979). Metacognition and cognitive monitoring: A new area of cognitive developmental enquiry. *American psychologist, 34*, 906-911.

Gray, S. S. (1991). Ideas in practice: Metacognition and mathematical problem solving. *Journal of Developmental Education*, 14, 24-28.

Halford, G. S. (1993). *Children's understanding: The development of mental models*. Hillsdale, NJ: Lawrence Erlbaum.

Hart, K. M. (1981). *Children's understanding of mathematics 11 - 16*. London: John Murray.

Karmiloff-Smith, A. (1979). Problem solving procedures of children's construction and representation of closed railway circuits. *Archives de Psychologie*, 47(180), 37-59.

Lave, J. (1993). Introduction. In S. Chaiklin & J. Lave (Eds.), *Understanding practice: perspectives on activity and context*. Cambridge University Press: Cambridge.

Lave, J. (1988). *Cognition in practice*. Cambridge: Cambridge University Press.

Lave, J., Smith, S., & Butler, M. (1988). Problem solving as an everyday practice. In R. I. Charles & E. A. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 61-81). Reston, VA: National Council of Teachers of Mathematics.

Lave, J., & Wenger, E. (1991). *Situated learning: legitimate peripheral participation*. Cambridge: Cambridge University Press.

Lester, F. K. (1985). Methodological considerations in mathematical problem solving instruction. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 41-70). Hillsdale, NJ: Lawrence Erlbaum.

Lester, F., & Kroll, D. L. (1990). Teaching students to be reflective: a study of two grade seven classes. *Proceedings of the 14th Psychology of Mathematics Education Conference PME-14, 1*, 151-158.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. London: Addison Wesley.

Mason, J., & Davis, J. (1991). *Fostering and sustaining mathematics thinking through problem solving*. Geelong, Victoria: Deakin University Press.

McGuiness, C. (1993). Teaching thinking: new signs for theories of cognition. *Educational Psychology, 13*(3&4), 305-316.

Neumark, V. (1995, September 8). For the love of maths. *Times Educational Supplement, 2* , 5.

Norman, D. A. (1986). Reflections on cognition and parallel distributed processing. In J. L. McClelland, D. E. Rumelhart, & PDP research group (Eds.), *Parallel distributed processing: explorations in the microstructure of cognition* (Vol. 2, pp. 531-546). Cambridge, Mass: MIT Press.

Nunes, T., Schliemann, A. D., & Carraher, D. W. (Eds.). (1993). *Street mathematics and school mathematics*. New York: Cambridge University Press.

Palincsar, A. S., & Brown, A. L. (1984). Reciprocal teaching of comprehension-fostering and monitoring activities. *Cognition and instruction, 1*(2), 177-185.

Perkins. (1987). Thinking frames: an integrated perspective on teaching cognitive skills. In J. B. Baron & R. J. Sternberg (Eds.), *Teaching thinking skills*. New York: W. H. Freeman.

Perkins, D., & Salomon, G. (1987). Transfer and teaching thinking. In D. Perkins, J. Lockhead, & J. Bishop (Eds.), *Thinking: the second international conference* (pp. 285-303). Hillsdale, NJ: Lawrence Erlbaum.

Prawat, R. (1989). Promoting access to knowledge, strategy and disposition in students: a research synthesis. *Review of Educational Research, 59*(1), 1-41.

Resnick, L. B. (1991). Shared cognition: Thinking as social practice. In L. B. Resnick, J. M. Levine, & S. D. Teasley (Eds.), *Perspectives on socially shared cognition* (pp. 1-22). Washington, DC: American Psychological Association.

Resnick, L. B. (1987). *Education and learning to think*. Washington, DC: National Academy Press.

Salomon, G., & Globerson, T. (1987). Skill may not be enough: The role of mindfulness in learning and transfer. *International Journal of Educational Research, 11*(6), 623-637.

Schoenfeld, A. H. (Ed.). (1994). *Mathematical thinking and problem solving*. Hillsdale, NJ: Lawrence Erlbaum.

Schoenfeld, A. H. (Ed.). (1987). *Cognitive science and mathematics education*. Hillsdale, N.J.: Lawrence Erlbaum.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. New York: Academic Press.

Siegler, R. S. (1991). *Children's thinking* (2nd ed.). Eaglewood Cliffs: Prentice Hall.

Silver, E. A. (1987). Foundations of cognitive theory and research for mathematics problem solving instruction. In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 33-61). Hillsdale, NJ: Lawrence Erlbaum.

Tanner, H. F. R., & Jones, S. A. (1993a). The development of metacognitive skills in mathematical modelling. In G. Wain (Ed.), *British Congress on mathematical Education: Research papers* (pp. 76-80). Leeds: University of Leeds.

Tanner, H. F. R., & Jones, S. A. (1993b). *Hands on maths*. Swansea: University of Wales Swansea.

Tanner, H. F. R., & Jones, S. A. (1994). Using peer and self assessment to develop modelling skills with students aged 11 to 16: A socio-constructive view. *Educational Studies in Mathematics*, 27(4), 413-431.

Tanner, H. F. R., & Jones, S. A. (1995a). *Better thinking, better mathematics*. Swansea: University of Wales Swansea.

Tanner, H. F. R., & Jones, S. A. (1995b). Teaching mathematical thinking skills to accelerate cognitive development. *Psychology of mathematics education conference (PME-19), Recife, Brazil, 3*, 121-128.

Literature on open-ended problems

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The bibliography given here is preliminary, in the sense that the literature is collected during several years from different resources by the author. There is no claim for completeness, and the author is happy to have more information (see address at the end).

The titles classified into the following groups, according to the different types of open problems explained in the introduction: 1. Investigations, 2. Open-Ended Problems, 3. Open Problems, 4. Problem Variations (Problem Fields, Problem Domains, Problems Without Question), 5. Problem Posing, 6. Practical Work, 7. Project Work, 8. Real-Life Situations, 9. General.

1. Investigations

ATM 1991. Points of Departure 1. An ATM Activity Book. Derby: ATM.
Curtis, T.P. 1975. Two sixth form investigations. Mathematics Teaching 73.
Edwards, C. 1974. A geo-board investigation. Mathematics Teaching 69.
Evans, J. 1987. Investigations – the state of the art. Math. School 16 (1), 27-30.
Favis, D. 1975. An investigation into rectangular numbers. Mathematics Teaching 71.
Irwin, C. 1976. A brief investigation into functions. Mathematics Teaching 75.
Jaworski, B. 1985. An Investigative Approach to Teaching and Learning Mathematics. Milton Keynes: The Open University.
Jaworski, B. 1994. Investigating Mathematics Teaching: A Constructivist Enquiry. Studies in Mathematics Education Series: 5. London: Falmer Press.
Lerman, S. 1989. Investigations: Where to now? In: Mathematics Teaching: The State of the Art (ed. P. Ernest). London: Falmer Press.
Mason, J. 1978. On investigations. Mathematics Teaching 84.
Morgan, C. 1994. Teachers assessing investigational mathematics: the role of "algebra". In: Proceedings of the eighteenth PME conference (eds. J.P. Ponte & J.F. Matos). Vol. III, 295–302. University of Lisboa, Lisboa (Portugal).
Pimm, D. 1992. "Why are we doing this?" Reporting back on investigations. In: Language in the Learning of Mathematics (ed. D. Sawada). Edmonton (Alberta): Arberta Teachers Association/Barnett House.
Pirie, S. 1987. Mathematical Investigations in Your Classroom. London: Macmillan.
Wells, D. 1993. Problem Solving and Investigations (3rd edition). Bristol: Rain Press.

Whitin, D.J. 1989. The Power of Mathematical Investigations. In: New Directions for Elementary School Mathematics (ed. P.R. Trafton), 183–190. NCTM 1989 Yearbook. Reston (VA): Council.

Wolf, A. 1990. Testing Investigations. In: Mathematics versus the National Curriculum (eds. P. Dowling & R. Noss). London: Falmer Press.

2. *Open-Ended Problems*

Anon. 1989. A Question of Thinking: A First Look at Students' Performance on Open-ended Questions in Mathematics. California Assessment Program (CAP). California State Department of Education. Sacramento (CA).

Borgensen, H.E. 1994. Open ended problem solving in geometry. Nordic Studies in Mathematics Education 2 (2), 6–35.

Clarke, D.J. & Sullivan, P.A. 1992. Responses to open-ended tasks in mathematics: characteristics and implications. In: Proceedings of the PME 16 conference (ed. W. Geeslin & K. Graham). Volume I, 137-144. Durham (NH): University of New Hampshire.

Grover, B.W. & Kenney, P.A. 1994. Preservice sencondary teachers' beliefs about mathematics and their expectations about student performance on open-ended assessment tasks. In: Proceedings of the sixteenth PME-NA conference (ed. D. Kirshner). Volume 2, 162–168. Baton Rouge (LA): Louisiana State University.

Hashimoto, Y. 1994. Theory and Practice in Mathematics Education Through Two Teaching Methods (Open-end Approach and Developmental Treatment of Mathematics Problems) in Japan. Background Papers for the ICMI Study Conference 8 may – 11 May 1994, 159–163. University of Maryland: College Park (MD).

Hashimoto, Y. & Sawada, T. 1984. Research on the mathematics teaching by developmental treatment of mathematical problems. In: Proceedings of the ICMI–JSME regional conference on mathematical education (ed. T. Kawaguchi), 309–313. Japan Society of Mathematical Education.

Iida, S. 1993. Value-laden, context-bounded and open-ended problem solving in mathematics education. In: Proceedings of the seventeenth PME conference (eds. I. Hirabayashi, N. Nohda, K. Shigematsu & F.-L. Lin). Vol. III, 17–24. University of Tsukuba, Tsukuba (Japan).

Nagasagi, E. & Becker, J. 1993. Classroom Assessment in Japanese Mathematics Education. In: Assessment in the Mathematics Classroom (ed. N.L. Webb) , 40–53. NCTM 1993 Yearbook. Reston (VA): Council.

Nohda, N. 1984. The heart of "open-approach" in mathematics teaching. In: Proceedings of ICMI-JSME regional conference on mathematical education (ed. T. Kawaguchi), 314–318. Tokyo: JSME.

Nohda, N. 1986. A study of "open-approach" method in school mathematics. Tsukuba Journal of Educational Study in Mathematics 5, 119–131.

Nohda, N. 1988a. Problem solving using "open-ended problems" in mathematics teaching. In: Problem Solving – A World View (eds. H. Burkhardt, S. Groves, A. Schoenfeld & K. Stacey), 225–234. Nottingham: Shell Centre.

Nohda, N. 1988b. Mathematical pattern-finding in elementary school, focus on pupils' strategies and difficulties in problem-solving. In: Proceedings of the PME-XII Conference (ed. A. Borbas). Volume II, 503–510. Veszprém.

Nohda, N. 1990a. Actual Communication in the Mathematical Classroom. In: Transforming Children's Mathematics Education. International Perspectives (eds. L.P. Steffe & T. Wood), 266–271. Hillsdale (NJ): Lawrence Erlbaum.

Nohda, N. 1990b. Paradigm of open-approach method in the mathematics classroom activities. Focus on mathematical problem solving. In: Proceedings of the PME-XII Conference (ed. A. Borbas). Volume II, 201–208. Veszprém.

Nohda, N. 1991. Paradigm of the "open-approach" method in mathematics teaching: Focus on mathematical problem solving. International Reviews on Mathematical Education [= ZDM] 23 (2), 32–37.

Nohda, N. 1995. Teaching and Evaluating Using "Open-Ended Problem" in Classroom. International Reviews on Mathematical Education [= ZDM] 27 (2), 57–61.

Pandey, T. 1991. A Sampler of Mathematics Assessments. Sacramento (CA): California Dept of Education.

Pehkonen, E. 1995a. Introduction: Use of Open-Ended Problems. International Reviews on Mathematical Education [= ZDM] 27 (2), 55–57.

Pehkonen, E. 1995b. Über Schülerreaktionen auf das Verwenden offener Probleme im Mathematikunterricht. Beiträge zum Mathematikunterricht 1995, 368–371. Verlag Franzbecker, Hildesheim.

Pehkonen, E. 1995c. On using open-ended problems in mathematics. Proceedings of the European Research Conference on the Psychology of Mathematics Education (ed. E. Cohors-Fresenborg), 66–69. University of Osnabrück (Germany).

Pehkonen, E. 1995d. On Pupils' Reactions to the Use of Open-ended Problems in Mathematics. Nordic Studies in Mathematics Education 3 (4), 43–57.

Shimada, S. (ed.) 1977. Open-end approach in arithmetic and mathematics – A new proposal toward teaching improvement. Tokyo: Mizuumishobo. [in Japanise]

Silver, E.A. & Adams, V.M. 1987. Using open-ended problems. Tips for Teachers. Arithmetic Teacher 34 (5), 34–35.

Szetela, W. 1991. Performance on open-ended problems. In: The 1990 Assessment of Mathematics in British Columbia (ed. D.F. Robitaille). Victoria (B.C.): Ministry of Education.

Wiliam, D. 1994. Assessing authentic tasks: alternatives to mark-schemes. Nordic Studies in Mathematics Education 2 (1), 48–68.

Williams, D. 1989. Assessment of open-ended work in the secondary school. In: Evaluation and Assessment in Mathematics Education (ed. D. F. Robitaille), 135–140. Science and Technology Education. Document Series 32. Paris: Unesco.

Wu, H. 1994. The Role of Open-Ended Problems in Mathematics Education. Journal of Mathematical Behavior 13 (1), 115–128.

3. Open Problems

Alseth, B. 1995. Bruk av åpne oppgaver i matematikkundervisningen. Tangenten 1, 18–26.

Krippner, D. 1977. Motivierung durch offene Aufgaben im Mathematikunterricht. Die Scholle, 613-623.

Mason, J. 1991. Mathematical problem solving: open, closed and exploratory in the UK. International Reviews on Mathematical Education [= ZDM] 23 (1), 14–19.

Pehkonen, E. 1988a. Open problem solving in mathematics teaching. Mathematics Teaching 123; 17,30.

Pehkonen, E. 1988b. Offene Aufgaben im Geometrieunterricht. Mathematiklehren, Heft 29, 16-19.

Pehkonen, E. 1991a. Introduction: Problem Solving in Mathematics. International Reviews on Mathematical Education 23 (1), 1–4.

Pehkonen, E. 1993. On Teachers' Criteria to Assess Mathematical Activities. In: Proceedings of the seventeenth PME conference (eds. I. Hirabayashi, N. Nohda, K. Shigematsu & F.-L. Lin). Volume I, 220–227. University of Tsukuba, Tsukuba (Japan).

Pehkonen, E. 1995. On pupils' reactions to the use of open-ended problems in mathematics. Nordic Studies in Mathematics Education 3 (4), 43–57.

Pehkonen, E. & Zimmermann, B. 1989. Offene Probleme im Mathematikunterricht. In: Mathematics Education Research in Finland. Yearbook 1987-88 (ed. P. Kupari), 55-77. Institute for Educational Research. Publication series B. Theory into Practice 39. University of Jyväskylä.

Silver, E. 1995. The Nature and Use of Open Problems in Mathematics Education: Mathematical and Pedagogical Perspectives. International Reviews on Mathematical Education [= ZDM] 27 (2), 67–72.

Stacey, K. 1991. Linking application and acquisition of mathematical ideas through problem solving. International Reviews on Mathematical Education [= ZDM] 23 (1), 8–14.

Stacey, K. 1995. The Challenges of Keeping Open Problem-Solving Open in School Mathematics. International Reviews on Mathematical Education [= ZDM] 27 (2), 62–67.

Stevenson, F.W. 1992. Exploratory Problems in Mathematics. NCTM Reston (Va.): Council.

Sullivan, P.A., Clarke, D.J. & Wallbridge, M. 1991. Problem solving with conventional mathematics content: Responses of pupils to open mathematical tasks. Australian Catholic University (Victoria). Mathematics Teaching & Learning Centre. Research Report 1.

Sullivan, P.A., Clarke, D.J., Spandel, U. & Wallbridge, M. 1992. Using content-specific open questions as a basis of instruction: A classroom experiment. Australian Catholic University (Victoria). Mathematics Teaching & Learning Centre. Research Report 4.

Zimmermann, B. 1991. Offene Probleme für den Mathematikunterricht und ein Ausblick auf Forschungsfragen. In: International Reviews on Mathematical Education [= ZDM] 23 (2), 38–46.

Zimmermann, B. (ed.) 1991. Problemorientierter Mathematikunterricht. Bad Salzdetfurth: Franzbecker.

4. Problem Variations (Problem Fields, Problem Domains, Problems Without Question)

Charles, R. & Lester, F. 1982. Teaching problem solving: What, why & how. Palo Alto (CA): Dale Seymour.

Kopka, J. 1994. Isolated and non-isolated problems. In: Selected Topics from Mathematics Education 3 (ed. R. Solvang), 5–20. Centre for Teacher Education and In-Service Training, University of Oslo.

Pehkonen, E. 1989a. Der Umgang mit Problemfeldern im Mathematikunterricht der Sek. I. In: Beiträge zum Mathematikunterricht 1989, 290–293. Verlag Franzbecker, Bad Salzdetfurth.

Pehkonen, E. 1989b. Verwenden der geometrischen Problemfelder. In: Geometry Teaching – Geometrieunterricht (ed. E. Pehkonen), 221-230. Department of Teacher Education. University of Helsinki. Research Report 74.

Pehkonen, E. 1990. Mathematische Problemfelder bieten andere Möglichkeit. Math. Unterr. Praxis 11 (1), 35-38.

Pehkonen, E. 1991b. Mathematical Problem Fields in the Classroom. In: Proceedings of the SEACME-5 (eds. P.K. Veloo, F. Lopez-Real & T. Singh), 253–258. Universiti Brunei Darussalam.

Pehkonen, E. 1992. Using Problem Fields as a Method of Change. The Mathematics Educator 3 (1), 3–6.

Pehkonen, E. 1993. On Teachers' Criteria to Assess Mathematical Activities. In: Proceedings of the seventeenth PME conference (eds. I. Hirabayashi, N. Nohda, K. Shigematsu & F.-L. Lin). Vol. 1, 220–227. University of Tsukuba, Tsukuba (Japan).

Pehkonen, E. & Zimmermann, B. 1990. Probleemakentät matematiikan opetuksessa ja niiden yhteys opetuksen ja oppilaiden motivaation kehittämiseen. Osa1: Teoreettinen tausta ja tutkimusasetelma. [Problem Fields in Mathematics Teaching and their Connection to the Development of Teaching and Pupils' Motivation. Part 1: Theoretical Background and Research Design.] University of Helsinki. Department of Teacher Education. Research Report 86.

Solvang, R. 1994. Thoughts about the free phase in connection to the use of generator problem. In: Selected Topics from Mathematics Education 3 (ed. R. Solvang), 56–67. Centre for Teacher Education and In-Service Training, University of Oslo.

5. *Problem Posing*

Brown, S.I. & Walter, M.I. 1990. The art of problem posing. Hillsdale (NJ): Lawrence Erlbaum (2nd edition).

Brown, S.I. & Walter, M.I. 1988. Problem posing in mathematics education. Questioning Exchange 2 (2), 123–131.

Brown, S.I. & Walter, M.I. (eds.) 1993. Problem posing: reflections and applications. Hillsdale (NJ): Lawrence Erlbaum.

Butts, T. 1980. Posing problems properly. In: Problem solving in school mathematics (eds. S. Krulik & R. Reys), 23–33. NCTM-Yearbook 1980. Reston (Va): Council.

Dillon, J.T. 1988. Levels of problem finding vs problem posing. Questioning Exchange 2 (2), 105–115.

Ellerton, N.F. 1986a. Children's made-up mathematics problems – A new perspective on talented mathematicians. Educational Studies in Mathematics 17, 261–271.

Ellerton, N.F. 1986b. Mathematics problems written by children. Research in Mathematics Education in Australia (December), 32–44.

Gonzales, N.A. 1994. Problem Posing: A Neglected Component in Mathematics Courses for Prospective Elementary and Middle School Teachers. School Science and Mathematics 94 (2), 78–84.

Kilpatrick, J. 1987. Problem Formulating: Where Do Good Problems Come From? In: Cognitive Science and Mathematics Education (ed. A.H. Schoenfeld), 123–147. Hillsdale (NJ): Lawrence Erlbaum.

Leung, S.S. 1993a. The relation of mathematical knowledge and creative thinking to the mathematical problem posing of prospective elementary school teachers on tasks differing in numerical information content. University of Pittsburgh (unpublished doctoral dissertation).

Leung, S.S. 1993b. Mathematical problem posing: the influence of task formats, mathematics knowledge, and creative thinking. In: Proceedings of the seventeenth PME conference (eds. I. Hirabayashi, N. Nohda, K. Shigematsu & F.-L. Lin). Vol. III, 33–40. University of Tsukuba, Tsukuba (Japan).

Leung, S.S. 1994. On analyzing problem-posing processes: a study of prospective elementary teachers differing in mathematics knowledge. In: Proceedings of the eighteenth PME conference (eds. J.P. Ponte & J.F. Matos). Vol. III, 168–175. University of Lisboa, Lisboa (Portugal).

Malara, N.A. & Gherpelli, L. 1994. Problem Posing and Hypothetical Reasoning in Geometrical Realm. In: Proceedings of the eighteenth PME conference (eds. J.P. Ponte & J.F. Matos). Vol. III, 216–223. University of Lisboa, Lisboa (Portugal).

Mamona-Downs, J. 1993. On analysing problem posing. In: Proceedings of the seventeenth PME conference (eds. I. Hirabayashi, N. Nohda, K. Shigematsu & F.-L. Lin). Vol. III, 41–48. University of Tsukuba, Tsukuba (Japan).

Moses, B., Bjork, E. & Goldenberg, E.P. 1990. Beyond problem solving: problem posing. In: Teaching and learning mathematics in the 1990s (ed. T.J. Cooney), 82–91. Reston (VA): NCTM.

Palhares, P.M.B. 1992. The introduction of a problem posing strategy as a means to teach mental arithmetic. Teses. Associação de Professores de Matemática. Universidad Lisboa.

Schwartz, J.L. 1991. Can We Solve the Problem Solving Problem Without Posing the Problem Posing Problem? Paper presented at the NATO Advanced Research Workshop: Information Technology and Mathematical Problem Solving Research. Viana do castelo, 27–30 April 1991.

Silver, E.A. 1993. On mathematical problem posing. In: Proceedings of the seventeenth PME conference (eds. I. Hirabayashi, N. Nohda, K. Shigematsu & F.-L. Lin). Vol. I, 66–85.. University of Tsukuba, Tsukuba (Japan).

Silver, E.A. 1994. On Mathematical Problem Posing. For the Learning of Mathematics 14 (1), 19–28.

Silver, E.A. & Mamona, J. 1989. Problem posing by middle school mathematics teachers. In: Proceedings of PME-NA 11 Conference (eds. C.A. Maher, G.A. Goldin & R.B. Davis). Volume 1, 263–269. New Brunswick (NJ): Rutgers University.

Silver, E.A. & Mamona, J. 1990. Stimulating Problem Posing in Mathematics Instruction through Open problems and "What-if-nots". In: Implementing New Curriculum and Evaluation Standards, 1990 PCTM (eds. G.W. Blume & M.K. Heid).

Silver, E.A., Mamona-Downs, J., Leung, S.S. & Kenney, P.A. 1996. Posing Mathematical Problems: An Exploratory Study. Journal for Research in Mathematics Education 27 (3), 293–309.

Skinner, P. 1991. What's your problem?: posing and solving mathematical problems, K–2. Portsmouth (NH): Heinemann.

Stoyanova,E. & Ellerton,N.F. 1996. A framework for research into students' problem posing in school mathematics. In: Technology in mathematics education (ed. P.Clarkson), 518–525. Melbourne: Mathematics Education Research Group of Australia.

Walter, M. 1987a. Generating problems from almost anything: Part 1. Mathematics Teaching 121, 3–7.

Walter, M. 1987b. Generating problems from almost anything: Part 2. Mathematics Teaching 121, 2–6.

Walter, M. 1988. Some roles of problem posing in the learning of mathematics. In: Mathematics, Teachers, and Children – A Reader (ed. D. Pimm), 190–200. London: Hodder and Stoughton.

Walter, M. 1989. Curriculum topics through problem posing. Mathematics Teaching 128, 23–25.

Walter, M.I. & Brown, S.I. 1977. Problem posing and problem solving: An illustration of their interdependence. Mathematics Teacher 70 (1), 4–13.

Yerushalmy, M. & Chazan, D. 1987. Effective problem posing in an inquiry environment: A case study using the Geometric Supposer. In: Proceedings of PME–NA 11 Conference (eds. C.A. Maher, G.A. Goldin & R.B. Davis). Volume 2, 53–59. New Brunswick (NJ): Rutgers University.

Yerushalmy, M., Chazan, D. & Gordon, M. 1993. Posing problems: One aspect of bringing inquiry into classroom. In: The Geometric Supposer: what is it a case of? (eds. J. Schwartz, M. Yerushalmy & B. Wilson), 117–142. Hillsdale (NJ): Erlbaum.

Zimmermann, B. 1986. From problem solving to problem finding in mathematics instruction. In: Mathematics Education Research in Finland. Yearbook 1985 (ed. P. Kupari), 81–103. Institute for Educational Research. Publication series B. Theory into Practice 3. University of Jyväskylä.

6. Practical Work

Jaworski, B. & Pimm, D. 1986. Practical Work in the Secondary Mathematics Classroom. Milton Keynes: The Open University.

7. Project Work

Leino, J. & al. 1989. Tietokone opetuksen kehittämisessä. 4. Projektiopiskelu koulussa. [Computers in development of teaching. 4. Project study in school.] Helsingin yliopisto. Kasvatustieteen laitos. Tutkimuksia 122.

Leino, J. 1991. Dynamic knowledge in school. University of Helsinki. Department of Education. Research Bulletin 79.

Leino, J. 1992. The importance of project work in teaching mathematics. In: Mathematics teaching through project work (ed. J. Leino). University of Tampere. Department of Teacher Education at Hämeenlinna. Publication nr. 27.

Leino, J. (ed.) 1992. Mathematics teaching through project work. University of Tampere. Department of Teacher Education at Hämeenlinna. Publication nr. 27.

Pehkonen, L. 1994. Project work – a way to learn actively. In: Constructivist Viewpoints for School Teaching and Learning in Mathematics and Science (eds. M. Ahtee & E. Pehkonen), 161–164. University of Helsinki. Department of Teacher Education. Research Report 131.

8. Real-Life Situations

Gravemeijer, K., van den Heuvel, M. & Streefland, L. 1990. Contexts, Free Productions, Tests, and Geometry in Realistic Mathematics Education. Utrecht University: OW&OC.

van der Heuvel–Panheizen, M. 1996. Assessment and realistic mathematics education. Utrecht University: Freudenthal institute.

Streefland, L. 1991. Fractions in Realistic Mathematics Education. A Paradigm of Developmental Research. Dordrecht: Kluwer Academic Publishers.

Streefland, L. (ed.) 1991. Realistic Mathematics Education in Primary School. Utrecht University: OW&OC.

9. General.

Ahtee, M. & Pehkonen, E. (eds.) 1994. Constructivist Viewpoints for School Teaching and Learning in Mathematics and Science. University of Helsinki. Department of Teacher Education. Research Report 131.

Anon. 1988. Mathematics for ages 5 to 16. Proposals of the Secretary of State for Education and Science and the Secretary of State for Wales. Department of education and Science and the Welsh Office.

Anon. 1990. Lehrplan Mathematik. Lehrplanrevision Gesamtschule. Sekundarstufe I. Behörde für Schule, Jugend und Berufsbildung, Amt für Schule. Hamburg.

van den Brink, J.F. 1987. Children as arithemetic book authors. For the Learning of Mathematics 7, 44–48.

Buth, M. 1989. Möglichkeiten und Grenzen des "offenen Unterrichts" – Folgerungen für die Mathematikdidaktik. In: Beiträge zum Mathematikunterricht 1989, 116-119. Bad Salzdetfurth: Verlag Franzbecker.

Cockcroft Report 1982. Mathematics counts. Report of the Committee of Inquiry into the Teaching of Mathematics in Schools. London: H.M.S.O.

Davis, R.B., Maher, C.A. & Noddings, N. (eds.) 1990. Constructivist Views on the Teaching and Learning of Mathematics. JRME Monograph Number 4. Reston (VA): NCTM.

Dörner, D., Kreuzig, H.W., Reither, F. & Städel, T. (Hrsg.) 1983. Lohausen. Vom Umgang mit Unbestimmtheit und Komplexität. Bern/ Stuttgart/ Wien: Huber.

Mason, J. 1982 (with L. Burton and K. Stacey). Thinking Mathematically. Bristol: Addison-Wesley.

Mason, J. & Davis, J. 1991. Fostering and sustaining mathematics thinking through problem solving. Geelong (Victoria): Deakin University.

NCTM 1989. Curriculum and Evaluation Standards for School Mathematics. Reston (Va.): Council.

NCTM 1991. Professional Standards for Teaching Mathematics. Reston (Va.): Council.

Pehkonen, E. (ed.) 1995. Using Open-Ended Problems in Mathematics. International Reviews on Mathematical Education [= ZDM] 27 (2).

Schoenfeld, A.H. 1984. Book review of Mason, J., Burton, L., Stacey, K.: Thinking mathematically. 1982. International Reviews on Mathematical Education (= ZDM) 16 (3), 103–105.

Silver, E.A., Kilpatrick, J. & Schlesinger, B. 1990. Thinking Through Mathematics. New York: The College Board.

Stoessiger, R. 1988. Using Language Learning Conditions in Mathematics. PEN from the primary english teaching association. Tasmania.

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